An analytic approach to the secular evolution of cataclysmic variables

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ABSTRACT

We investigate the reaction of low–mass stars to mass loss in the context of cataclysmic variables (CVs). Based on homology we derive a first order differential equation for the radius reaction of a low–mass star upon mass loss. The solution of the differential equation yields the stellar radius as a function of the mass–transfer time–scale $\tau_{\mathrm{M}} = |M/\dot{M}|$ and the stellar mass M. The main property of the differential equation is the convergence of solutions which differ only in the initial conditions. The solutions converge on an e–folding time–scale τ_{per}. A linearized analysis yields $\tau_{\mathrm{per}} \lesssim 1/20\,\tau_{\mathrm{KH}}$ (with τ_{KH} the Kelvin–Helmholtz time of the star). Applying our model to CV evolution we furthermore show that after a short turn–on phase the CV evolution is independent of its initial conditions, which explains the similarity of secular evolution tracks previously found by Paczyński & Sienkiewicz and Kolb & Ritter in numerical studies. It is therefore why the detached phase ($\dot{M} = 0$) in the life of a single CV can still be seen in the whole CV population as a period gap. An analytic solution of the linearized differential equation near thermal equilibrium is applied to the period flag subsequent to the turn–on of mass–transfer after the detached phase of CV evolution.

Key words: stars: mass-loss – novae, cataclysmic variables

1 INTRODUCTION

Cataclysmic variables (CVs) are short–period semidetached binaries consisting of a white dwarf (WD) primary and a low–mass companion (the secondary) which fills its critical Roche lobe and loses mass through the inner Lagrangian point L_1. The transferred matter is finally accreted by the white dwarf where, on average, the energy liberated by the accretion process dominates all the other possible sources in luminosity (e.g. Warner 1995).

The observed distribution of the orbital periods of CVs is bimodal with many systems in the period ranges 80min $\lesssim$ P $\lesssim$ 120min and 3h $\lesssim$ P $\lesssim$ 16h but very few of them in the range 2h $\lesssim$ P $\lesssim$ 3h (Ritter & Kolb 1995). The dearth of systems in the 2...3h period range is usually referred to as the period gap.

The now canonical explanation for the occurrence of the gap is the model of disrupted magnetic braking (Rappaport, Verbunt & Joss 1983; Spruit & Ritter 1983). According to this model a CV detaches when the oversized secondary becomes fully convective (at $P = P_{\mathrm{u}} \simeq$ 3h) and subsequently evolves as a detached binary without mass–transfer (and therefore virtually unobservable) until mass–transfer resumes at $P = P_{\mathrm{l}} \simeq 2$h (for a review see, e.g., King 1988).

A remarkable feature of the period gap is that its boundaries at P_{u} and P_{l} are welldefined.

Since the gap is a collective phenomenon of the whole CV population, the secular evolution (i.e. the long–term evolution) of the vast majority of CVs with periods $P \gtrsim P_{\rm u} \simeq 3$h must necessarily be very similar, otherwise the coherence of the phenomenon 'period gap' would be lost. For the occurrence of the sharp boundary at $\sim$ 2h all the companions within the detached phase of the period gap have to be of the same mass. The definite boundary at $\sim$ 3h then suggests that all companions are equally oversized when they switch off mass–transfer.

Numerical calculations of the secular evolution of CVs with different initial parameters show indeed that the corresponding evolutionary tracks fulfil these requirements and are at least near $P \simeq$ 3h very similar. Already Paczyński & Sienkiewicz (1983) found that evolutionary tracks calculated for a given total mass but different values for the initial secondary mass converged very rapidly to a single track. Later and more detailed calculations (see, e.g., Kolb & Ritter 1992, and references therein) confirmed this result.

It is the aim of this paper to show analytically why the evolutionary tracks tend to converge on a rather short time scale on the order of only a few 10^6yr. Use of homology approximation yields a description of the reaction of the secondary star upon mass loss. We derive a solvable first–order differential equation for the stellar radius as a function of mass and mass loss rate. The fundamental property of this differential equation is the convergence of any two solutions which differ only in the initial conditions on a rather short times–cale which, in turn, explains the similarity of the evolutionary tracks.

This paper is organized as follows. In Section 2 we derive the basic differential equation whose fundamental properties are then investigated in Section 3. In Section 4 we discuss briefly a number of applications to the secular evolution of CVs. In Section 5 we integrate the differential equation numerically and compare the results with those of our analytic approximation. Our main conclusions are summarized in Section 6.

2 THE REACTION OF PREDOMINANTLY CONVECTIVE LOW–MASS MAIN SEQUENCE STARS UPON MASS LOSS

Since for stars in thermal equilibrium the specific entropy $s_{\rm e}(M_{\rm r}, M)$ (where the subscript e refers to thermal equilibrium) at a given mass shell $M_{\rm r}$ depends on the total mass M, mass loss drives the star out of thermal equilibrium (i.e. $s(M_{\rm r}, M) \neq s_{\rm e}(M_{\rm r}, M)$). Therefore, the radius R of a star undergoing mass loss is, in general, different from its equilibrium value ($R \neq R_{\rm e}$). Thermal relaxation occurs on the thermal time–scale $\tau_{\rm th}$ which is of the order of the Kelvin–Helmholtz time $\tau_{\rm KH} = GM^2/RL$. Here L is the star's luminosity, and G the gravitational constant.

2.1 Slow and rapid mass loss

In the limit of slow mass loss ($\tau_{\rm th} \ll \tau_{\rm M} = |M/\dot{M}|$), the star thermally relaxes to $s(M_{\rm r}, M) \simeq s_{\rm e}(M_{\rm r}, M)$. Therefore $R \simeq R_{\rm e}$ and $\zeta_{\rm eff} \simeq \zeta_{\rm e}$, where $\zeta_{\rm eff} = ({\rm d}\ln{\rm R}/{\rm d}\ln{\rm M})_{\rm eff}$ is the effective mass–radius exponent of the mass losing star, and $\zeta_{\rm e}$ is the mass–radius exponent in thermal equilibrium, i.e. in our case, the mass–radius exponent for main sequence stars.

For very rapid mass loss ($\tau_{\rm M} \ll \tau_{\rm th}$) the specific entropy remains essentially unchanged, i.e. $s(M_{\rm r} < M, M) \simeq s(M_{\rm r} < M, M_{\rm i})$, where $M_{\rm i}$ is the initial mass. In this case, $\zeta_{\rm eff}$ is equal to the adiabatic mass radius exponent $\zeta_{\rm ad}$. The adiabatic reaction of low–mass stars to mass loss has been studied in detail by Hjellming & Webbink (1987) and Hjellming (1989). In the context of this paper, it is important that $\zeta_{\rm ad}$ is mainly a function of the relative mass of the convective envelope and that for predominantly convective stars $\zeta_{\rm ad} = -1/3$, whereas for mainly radiative ones $\zeta_{\rm ad}$ is a large positive number.

2.2 Mass loss on a thermal time–scale

The intermediate case of $\tau_{\rm M} \sim \tau_{\rm th}$ has been treated by Lauterborn & Weigert (1972) in an analytical approximation. They integrated the gravito–thermal energy generation $\epsilon_{\rm g}$ over the convective envelope of the star, for which they assumed a polytrope of index n=3/2 and where the entropy in the convective envelope is determined from a radiative grey atmosphere. For the opacity they assumed a power–law approximation

$$\kappa = \kappa_0 P^{\rm a} T^{\rm b}, \tag{1}$$

where P and T are respectively the pressure and the temperature. Starting from Lauterborn & Weigert's result and the fact that in the case of rapid mass loss ($\tau_{\rm M} \to 0$) the result is known, i.e. $\zeta_{\rm eff} = \zeta_{\rm ad}$, D'Antona, Mazzitelli & Ritter (1989) derived the following expression for the reaction of a predominantly convective star upon mass loss:

$$\zeta_{\rm eff} = \zeta_{\rm ad} + c \left(\frac{L_{\rm g}}{L_{\rm e}}\right) \left(\frac{R}{R_{\rm e}}\right) \frac{\tau_{\rm M}}{\tau_{\rm KH,e}}, \tag{2}$$

where the subscript e denotes again values in thermal equilibrium. $L_{\rm g}$ is the gravo–thermal luminosity and $c = 1.75\,(a{+}1)$ with $a = (\partial \ln \kappa / \partial \ln P)_{\rm T}$ the pressure exponent of the opacity law (1). Since the relevant source of opacity is $\rm H^-$, typical values are $a \simeq 0.5$, $b \simeq$4...5, and thus $c \simeq 2.6$.

For calculating the radius R of the donor in a mass–transferring close binary system one has to integrate (2) over the whole mass–transfer history. However, for this $L_{\rm g}$, which is undetermined so far, has to be known explicitly.

2.3 Basic equations of the model

In the following we shall derive an explicit expression for $L_{\rm g}$. For this we make use of homology relations and of the same approximations on which equation (2) is founded. Our procedure is adequate for predominantly convective stars.

In order to transform (2) into an equation with the radius R as the only unknown variable, we have to express $L_{\rm g}$ as a function of R. If we write the unknown $L_{\rm g}$ as

$$L_{\rm g} = L(R) - L_{\rm nuc}(R) \tag{3}$$

the problem of determining $L_{\rm g}$ is shifted to determining the surface luminosity $L(R)$ and the nuclear luminosity $L_{\rm nuc}(R)$. Although it is not clear a priori that L and $L_{\rm nuc}$ can be expressed as functions of R alone, we shall show that, at least within the approximations used here, this is in fact possible.

We derive $L_{\rm nuc}(R)$ from homology relations (e.g. Kippenhahn & Weigert 1994). Writing

$$\epsilon_{\rm nuc} \sim \rho T^{\nu}, \qquad \rho \sim \left(\frac{M}{R^3}\right), \qquad T \sim \left(\frac{M}{R}\right), \tag{4}$$

where ρ is the density, one obtains

$$L_{\rm nuc}(R) = L_{\rm e} \left(\frac{R}{R_{\rm e}}\right)^{-(3+\nu)}. \tag{5}$$

Since the nuclear energy generation process relevant for low–mass stars is the pp-chain, ν is typically of the order of $\nu \sim$ 4...6. However, ν can be larger if the star is inflated and has therefore a lower central temperature, and because nuclear burning via the ppI–chain is incomplete. We demonstrate that in Fig. 1 where we plot ν_ρ as a function of temperature for a constant density of $\rho = 150$g cm^{-3} which is typical for the central density of a low–mass star. The nuclear energy generation rates are taken from Fowler, Caughlan & Zimmermann (1975) and Harris et al. (1983). Fig. 1 shows clearly the increase of ν_ρ to lower temperatures.

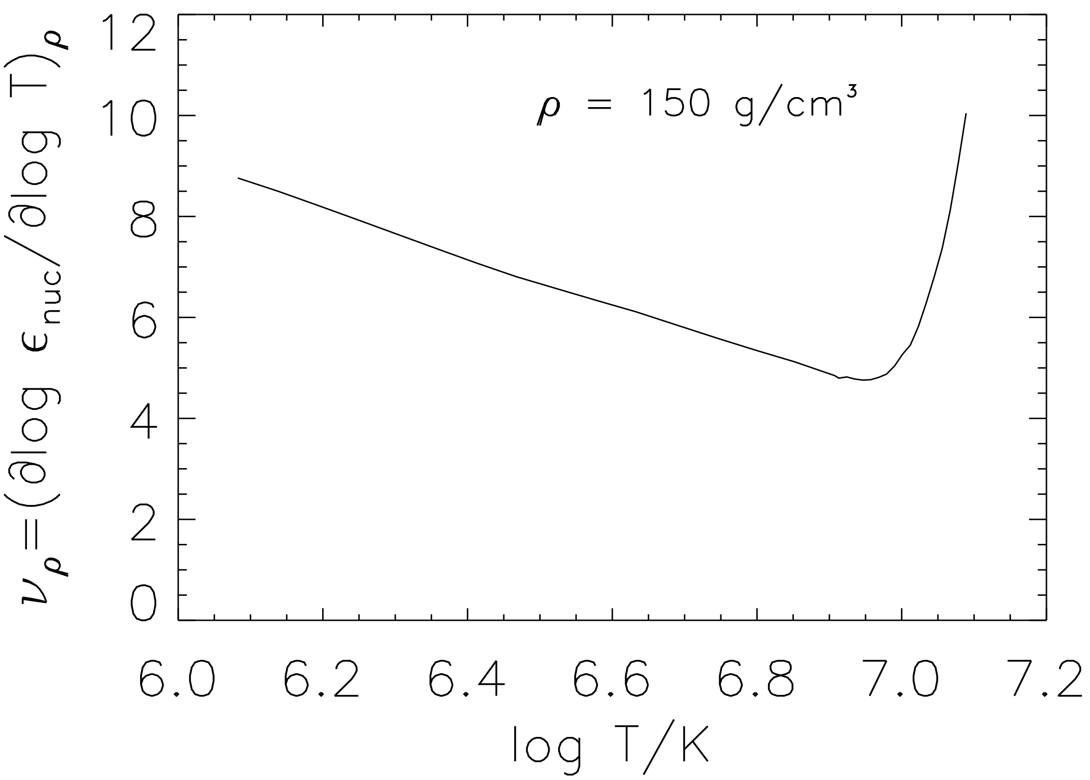

Figure 1. The run of $\nu_\rho = (\partial \log \epsilon_{\rm nuc}/\partial \log T)_\rho$ with temperature at a density $\rho = 150$ g cm^{-3}, a value typical for the centre of low–mass stars. $\epsilon_{\rm nuc}$ has been taken from Fowler et al. (1975) and Harris et al. (1983).

The surface luminosity $L(R)$ is given by the Stefan–Boltzmann law

$$L(R) = 4\pi\sigma\, R^2 T_{\rm eff}^4(R), \tag{6}$$

where we determine the effective temperature $T_{\rm eff}$ from the analytic theory of the Hayashi line (e.g. Kippenhahn & Weigert 1994). This yields

$$T_{\rm eff}(R) = T_{\rm eff,e}\ \left(\frac{R}{R_{\rm e}}\right)^{\mu}, \tag{7}$$

with

$$\mu := \frac{(3-n)\ (a+1)\ -\ 2}{(n+1)\ (a+1)\ +\ b}, \tag{8}$$

and a and b are defined via equation (1), hence $|\mu| \ll 1$. Here n is the polytropic index of the convective envelope.

Having now determined $L(R)$ and $L_{\rm nuc}(R)$ we combine equations (2), (3), (5), (6) and (7) to write $\zeta_{\rm eff}$ as a function of R only. It will turn out that it is much more convenient to discuss the problem in the dimensionless variable $r = R/R_{\rm e}$ instead of R. We point out, however, that beside R also the scaling factor $R_{\rm e}$ depends on M.

Doing so, we obtain

$$\frac{{\rm d}\ln r}{{\rm d}\ln M} = c\frac{\tau_{\rm M}}{\tau_{\rm KH,e}} H(r,M), \tag{9}$$

where the abbreviation H stands for

$$H = r^{(3+4\mu)} - r^{-(2+\nu)} - \Gamma \tag{10}$$

and Γ for

$$\Gamma = \frac{1}{c}\frac{\tau_{\mathrm{KH,e}}}{\tau_{\mathrm{M}}}\left(\zeta_{\mathrm{e}} - \zeta_{\mathrm{ad}}\right). \tag{11}$$

To close the set of equations, we have to specify τ_{M}. One application of our model is the loss of mass owing to a strong wind. Then $\dot{M}$ is specified by the physics of the wind and equation (9) describes the radius reaction for this given wind mass loss.

The standard model for CV evolution assumes that the loss of orbital angular momentum $\dot{J}$ necessary to drive mass–transfer in CVs is due to magnetic braking and gravitational wave radiation. Magnetic braking is thought to operate only for donors with a radiative core. Stationary mass–transfer where the donor fills its Roche lobe ($R = R_{\mathrm{R}}$ and $\dot{R} = \dot{R}_{\mathrm{R}}$; R_{R} is the Roche radius) demands

$$\tau_{\mathrm{M}} = \frac{1}{2}\left(\zeta_{\mathrm{eff}} - \zeta_{\mathrm{R}}\right)\tau_{\mathrm{J}}\,, \tag{12}$$

where $\tau_{\mathrm{J}} = |J/\dot{J}|$ denotes the time–scale of orbital angular momentum loss, and $\zeta_{\mathrm{R}} = \partial \ln R_{\mathrm{R}}/\partial \ln M$ the mass–radius exponent of the critical Roche radius (see, e.g., Ritter 1988). τ_{J} is generally thought to be a function of M as well as of R. Thus equations (9) – (11) have to be solved simultaneously with equation (12). However, since there is no formulation of τ_{J} from first principles, and due to the insensitivity in the present formulation of magnetic braking (like that of Verbunt & Zwaan 1981) and gravitational radiation upon $\delta R = (R - R_{\mathrm{e}}) \ll R$, we simplify the problem by assuming $\tau_{\mathrm{J}} = \tau_{\mathrm{J}}(R_{\mathrm{e}}, M)$. We will show in Section 3 that for most parts of standard CV evolution $\zeta_{\mathrm{eff}} \simeq \zeta_{\mathrm{e}}$ and therefore the assumption $\tau_{\mathrm{M}} = \tau_{\mathrm{M}}(M)$ is reasonable. This is confirmed in numerical studies by Kolb (1992, 1993) and Hameury et al. (1988, 1991). By assuming that $\tau_{\mathrm{M}}(M)$ is a function of M only, we are incapable to treat problems where $\dot{M}$ is strongly influenced by radius changes $\dot{R}$ as this is e.g. the case when the the secondary is irradiated by the accretion luminosity of the transfered mass (King et al. 1995, Ritter, Zhang & Kolb 1996).

To solve equation (9) we have to specify initial conditions. Ritter (1988) found an exponential increase of $\dot{M}$ on a time–scale $\tau_{\dot{\mathrm{M}}} \simeq$ few 10^4yr when the secondary begins to fill its Roche lobe. As the total mass lost during this turn–on phase of mass–transfer is negligible (see Ritter 1988) the secondary is still very close to thermal equilibrium. We therefore use, without any restrictions, $R_{\mathrm{i}} = R_{\mathrm{e}}(M_{\mathrm{i}})$ and $\tau_{\mathrm{M,i}} = \tau_{\mathrm{M}}(M_{\mathrm{i}})$ as initial conditions.

Formulating the problem in this way, the reaction of predominantly convective stars upon mass loss depends only on τ_{M} and on properties of the star in thermal equilibrium. The latter can be assumed to be well known from numerical models (e.g. Van den Berg et al. 1983, Neece 1984, D'Antona & Mazzitelli 1982, 1985). Therefore, if the mass–transfer rate is specified, equation (9) can be integrated. We do this in section 5.

Although in deriving equation (9) we have used relations which are valid only for predominantly convective stars, the applicability of equation (9) is not necessarily restricted to such stars. In fact, as we shall show in section 5, equation (9) reproduces the results of full stellar evolution calculations quite well even when applied to stars in the mass range $0.7\,M_\odot \lesssim M \lesssim 1\,M_\odot$ which have relatively thin convective envelopes. That equation (9) works so well in these cases must mean that on the one hand thermal relaxation of these stars is well described by a relation of the form of equation (2), and that on the other hand $\mu > -3/4$ such that in equation (10) the exponents of the r terms have opposite signs.

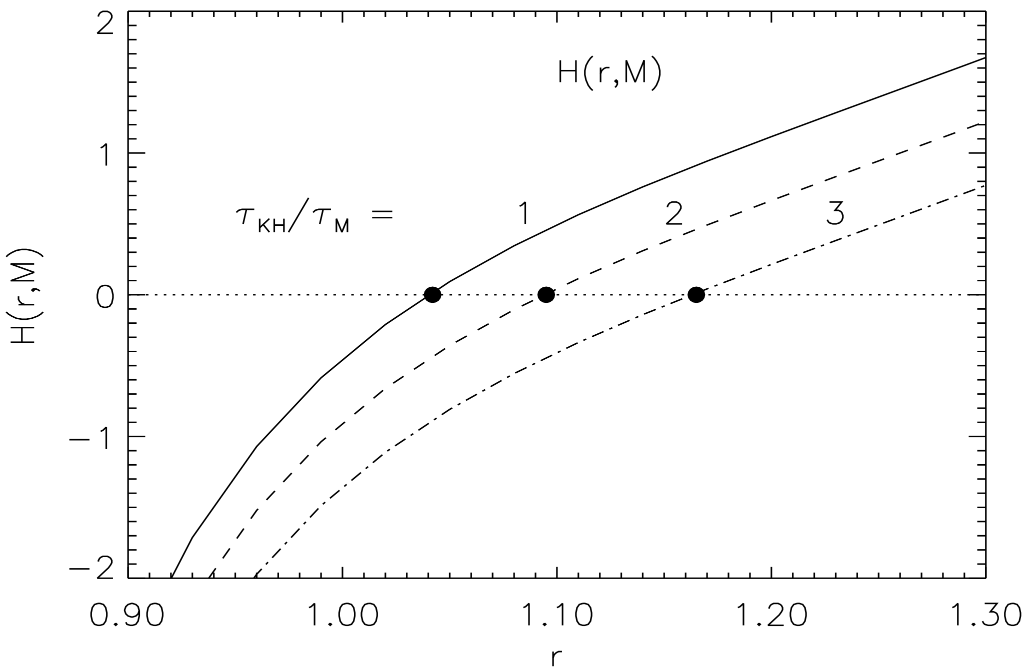

Figure 2. The run of the right–hand side of the differential equation (9) $H(r, M)$ with r for three different values of Γ, i.e. of $\tau_{\mathrm{KH,e}}/\tau_{\mathrm{M}}$. For this illustration we have chosen $(\zeta_{\mathrm{e}} - \zeta_{\mathrm{ad}})/c = 0.43$, valid for fully convective stars. The local zero–points $H = 0$ are marked with filled circles.

3 ANALYTIC PROPERTIES OF THE MODEL

3.1 Limiting cases

At the onset of mass loss (e.g., in CVs) the donor star is in thermal equilibrium ($R = R_{\mathrm{e}}$, $r = 1$) and equation (9) yields $\zeta_{\mathrm{eff}} = \zeta_{\mathrm{ad}}$. If $\zeta_{\mathrm{e}} \lesseqgtr \zeta_{\mathrm{ad}}$ we find $\mathrm{d}\ln r/\mathrm{d}\ln M \gtreqless 0$, or, to put it another way, evolution with mass loss results in a star which is oversized ($\zeta_{\mathrm{ad}} < \zeta_{\mathrm{e}}$) or undersized ($\zeta_{\mathrm{ad}} > \zeta_{\mathrm{e}}$) compared to the thermal equilibrium radius.

To treat the case of rapid mass loss, we rewrite equation (9) in the form

$$\zeta_{\mathrm{eff}} = \zeta_{\mathrm{ad}} + c\frac{1}{\tau_{\mathrm{KH,e}}}\left(r^{(3+4\mu)} - r^{-(2+\nu)}\right)\ \tau_{\mathrm{M}}. \tag{13}$$

In the limit $\tau_{\mathrm{M}} \to 0$ $\zeta_{\mathrm{eff}} = \zeta_{\mathrm{ad}}$, as expected. For finite τ_{M} ζ_{eff} is given by ζ_{ad} plus a correction term which is proportional to τ_{M}.

In the case of slow mass–transfer ($\tau_{\mathrm{M}} \to \infty$) equation (9) shows that ζ_{eff} remains finite only if $r = 1$, i.e., if $R = R_{\mathrm{e}}$ and $\zeta_{\mathrm{eff}} = \zeta_{\mathrm{e}}$.

3.2 The case Γ =constant

For the special case of Γ =constant and $c\tau_{\mathrm{M}}/\tau_{\mathrm{KH,e}}$ =constant the right–hand side of equation (9) is independent of M, and thus equation (9) reduces to a non–linear autonomous differential equation (see, e.g., Bender & Orszag 1978). The stationary solution $r_0 = (R_0/R_{\mathrm{e}})$ given by

$$\left.\frac{\mathrm{d}\ln \mathrm{r}}{\mathrm{d}\ln \mathrm{M}}\right|_{r_0} = 0. \tag{14}$$

is a stable node (Bender & Orszag 1978). To illustrate this, we show, in Fig. (2) $H(r, M)$ as a function of r for different values of Γ. $H(r, M)$ is a monotonically increasing function of r with $H \to -\infty$ as $r \to 0$ and $H \to +\infty$ as $r \to +\infty$. If therefore e.g. $r > r_0$ (i.e. $R > R_0$) then $H(r, M) > 0$ and $\mathrm{d}\ln\mathrm{r} < 0$ since with mass loss $\mathrm{d}\ln\mathrm{M} < 0$. Thus, if $R > R_0$ the radius R will decrease toward $R = R_0$ until $H = 0$ is reached. In the mass–radius diagram the solution $R_0(M)$ proceeds with a constant offset $\ln(R_0/R_\mathrm{e})$ 'parallel' to the thermal equilibrium mass–radius relation, i.e. to the main sequence, and for this solution $\zeta_\mathrm{eff} = \zeta_\mathrm{e}$. r_0 can be obtained from $H(r_0, M) = 0$ which reads

$$r_0^{(3+4\mu)} - r_0^{-(2+\nu)} - \Gamma = 0. \tag{15}$$

For the secondaries in CV evolution entering the period gap $R_0 \simeq 1.25 R_\mathrm{e}$.

The fact that R_0 is a stable node shows that all evolutions with different initial conditions tend to converge to a uniform evolution which is asymptotically given by the stationary solution R_0. We note, however, that within a finite stellar mass change only those evolutions with initial conditions sufficiently close to each other will converge to one evolution. In the next section we compute the time–scale for the global convergence of solutions for the general case $\Gamma \neq$constant This enables us to specify the domain of initial conditions leading to solutions which converge within a fraction of τ_M to one uniform solution.

3.3 The general case

We now turn to the more general case where $\Gamma = \Gamma(M)$ is a specified function of the secondary mass. We show the convergence of solutions of equation (9) with different initial conditions for a given function $\Gamma(M)$ using perturbation methods. Consider a solution of equation (9), $r_\mathrm{ref} = R_\mathrm{ref}/R_\mathrm{e}$, hereafter called reference solution, and a solution of (9) corresponding to a small deviation from the reference solution,

$$R = R_\mathrm{ref} + \delta R, \tag{16}$$

where we assume $|\delta R| \ll R_\mathrm{ref}$. After a transformation to the variable $t = t(M)$ which is bijective as long as $\dot{M} \neq 0$, we linearize (9) in $\delta R/R_\mathrm{e}$ around $R_\mathrm{ref}/R_\mathrm{e}$ and obtain

$$\frac{\mathrm{d}\ln(\delta R/R_\mathrm{ref})}{\mathrm{d}t} \simeq -\gamma. \tag{17}$$

The function $\gamma = \gamma(M)$ depends only on the reference solution and on values of the corresponding thermally relaxed star, both of which are thought to be known:

$$\gamma = \frac{c}{\tau_\mathrm{KH,e}} \left\{ (3+4\mu) r_\mathrm{ref}^{(3+4\mu)} + (2+\nu) r_\mathrm{ref}^{-(2+\nu)} \right\}. \tag{18}$$

If γ remains reasonably constant in time, and this is in fact the case as we shall show below, we can integrate (17) to yield

$$\left(\frac{\delta R}{R_\mathrm{ref}}\right)_\mathrm{f} = \left(\frac{\delta R}{R_\mathrm{ref}}\right)_\mathrm{i} \exp(-\gamma\, t) \tag{19}$$

where δR_f is the perturbation at time t after an initial deviation δR_i. As long as $\mu > -3/4$ and $\nu > -2$, i.e. $\gamma > 0$, the perturbation δR decreases on the e–folding time $\tau_\mathrm{per} = 1/\gamma$. Now, as can easily be seen from equation (18), γ as a function of r_ref is minimal for

$$r_\mathrm{ref} = \left(\frac{2+\nu}{3+4\mu}\right)^{2/(5+4\mu+\nu)}. \tag{20}$$

With $\mu \simeq 0$, $\nu \simeq 6$ and $c \simeq 2.6$ we find that the minimum of γ is $\gamma_\mathrm{min} \simeq 7c/\tau_\mathrm{KH,e}$ and thus

$$\tau_\mathrm{per} = \frac{1}{\gamma} < \frac{1}{\gamma_\mathrm{min}} \simeq \frac{1}{18}\tau_\mathrm{KH,e}. \tag{21}$$

The fact that $\tau_{\rm per}$ is so short justifies our above assumption that $\gamma \simeq$ constant over the time of interest. Thus we have shown the convergence of solutions of equation (9) and, what is more, that the time–scale $\tau_{\rm per}$ on which an initial deviation from a reference path decays is small compared to the Kelvin–Helmholtz time. Since all initial deviations $\delta R_{\rm i}$ from the reference path $R_{\rm ref}$ decay on the short time–scale $\tau_{\rm per}$ $R_{\rm ref}$ is an attracting solution. In fact, since the only condition for $R_{\rm ref}$ to be an attracting solution was that $R_{\rm ref}$ is a solution of the differential equation (9) all solutions of (9) are attracting solutions. In the mass–radius diagram evolutions with different $R_{\rm i}(M_{\rm i})$ tend to bunch to one radius evolution as mass decreases and the initial spread in radius decays on the e–folding time $\tau_{\rm per}$. This in turn implies that solutions of (9) starting from arbitrary initial conditions converge assymptotically to a uniform evolution. Put it another way: except for effects owing to nuclear evolution, the donor star 'forgets' the initial conditions at the onset of mass–transfer after only a few $\tau_{\rm per}$ and the subsequent evolution is independent of the initial conditions.

In the case of the secular evolution of CVs during which $\tau_{\rm M} \gtrsim 0.25\tau_{\rm KH,e}$ most of the time, $\tau_{\rm per}$ is also small compared to or at most comparable to the mass–transfer time–scale. In CV evolution the secondaries are typically inflated by 25 per cent compared to $R_{\rm e}$ when they enter the period gap. They reach this radius within $0.01R_{\rm e}$ in $3\tau_{\rm per}$ after the turn on of mass–transfer. This explains in a natural way the convergence of the evolutionary tracks starting from different initial secondary masses found in numerical calculations of CV evolution.

As long as $\Gamma(M)$ does not change rapidly, the stellar structure of the secondary is then a function only of the momentary mass and mass loss rate. Therefore, when the CVs reach the period gap, those of them which had a sufficiently high initial secondary mass have the same orbital period by the time the systems enter the detached phase. It is for this reason that the period gap in the period distribution of the observed CVs is a prominent phenomenon.

3.4 Approximations for the uniform evolution

In this section we are interested to find analytic expressions for the evolution after the turn on phase. As pointed out in Section 3.3 all solutions of equation (9) are attracting solutions and represent after some $\tau_{\rm per}$ also all those solutions which were initially close by. To find this uniform evolution, we thus have to find one solution of equation (9) with a sufficiently high initial stellar mass.

For $\Gamma(M)$ =constant we know that $R_0(M)$ is the stationary solution of equation (9); hence the uniform solution is derived by $H(r_0, M) = 0$. For $\Gamma \neq$constant we can still approximate the uniform evolution by $r_0(M)$, the solution of $H(r_0(M), M) = 0$ for a given $\Gamma(M)$, as long as the dependence of $\Gamma(M)$ on M is sufficiently weak. For a quantitative analysis we define the time–scale τ_0 as

$$\frac{1}{\tau_0} = \left|\frac{\rm d}{\rm dt}\left(\frac{R_0}{R_{\rm e}}\right)\right| = \frac{1}{\tau_{\rm M}}\left|\frac{{\rm d}r_0}{\rm d\ln M}\right| \tag{22}$$

on which r_0 changes with mass because of $\Gamma(M)$. As long as $\tau_{\rm per} \ll \tau_0$ the uniform evolution is well approximated by r_0.

The time–scale τ_0 follows from dH = 0 which gives

$$\tau_0 = \frac{\tau_{\rm M}}{\tau_{\rm KH,e}} f(r_0)\left|\frac{{\rm d}\Gamma}{\rm d\ln M}\right|^{-1}\tau_{\rm KH,e}, \tag{23}$$

where we abbreviated

$$f(r_0) = (3+4\mu)r_0^{(2+4\mu)} + (2+\nu)r_0^{-(3+\nu)}. \tag{24}$$

We estimate τ_0 by using in equation (23) the minimum of $f(r_0)$ which gives $f_{\rm min} \simeq 5.8$ for

$\nu = 6$ and $\mu = 0$. Therefore r_0 is still a good approximation of the uniform evolution as long as

$$\left|\frac{\mathrm{d}\Gamma}{\mathrm{d}\ln \mathrm{M}}\right| \ll \frac{f_{\mathrm{min}}}{\tau_{\mathrm{per}}}\left(\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}\right)\tau_{\mathrm{KH,e}} \simeq 100\left(\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}\right), \tag{25}$$

which is indeed fulfilled in most cases of interest.

3.5 Evolution near thermal equilibrium for small $|\dot{M}|$

Let us now consider the evolution of a CV near the thermal equilibrium of the secondary star where $R(M) \simeq R_{\mathrm{e}}(M)$ with the initial stellar radius $R(M_{\mathrm{A}}) = R_{\mathrm{e}}(M_{\mathrm{A}})$ at the initial mass M_{A}.

We seek to find solutions of equation (9) for sufficiently small values of $|\dot{M}|$ so that $\Gamma \ll r \simeq 1$. In this case a small parameter $\epsilon = \Gamma$ is explicitly introduced in the problem about which we are allowed to perform a perturbation expansion of the form (Nayfeh 1973)

$$r = \widehat{r_0} + \epsilon\widehat{r_1} + \epsilon^2\widehat{r_2} + \ldots \tag{26}$$

The initial condition is fulfilled by $r(M_{\mathrm{A}}) = \widehat{r_0}(M_{\mathrm{A}}) = 1$ and $\widehat{r_n}(M_{\mathrm{A}}) = 0$ for $n > 0$. Equation (9) now reads

$$\frac{\mathrm{d}r}{\mathrm{d}\ln M} = c\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}\left\{r^{(4+4\mu)} - r^{-(1+\nu)} - \epsilon r\right\}. \tag{27}$$

After substituting the perturbation expansion (26) into equation (27), each coefficient of ϵ must vanish independently. We thus find for $\widehat{r_0}$

$$\frac{\mathrm{d}\widehat{r_0}}{\mathrm{d}\ln M} = c\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}\left(\widehat{r_0}^{(4+4\mu)} - \widehat{r_0}^{-(1+\nu)}\right) \tag{28}$$

with the solution $\widehat{r_0} = 1$ and using this result for $\widehat{r_1}$

$$\frac{\mathrm{d}\widehat{r_1}}{\mathrm{d}\ln M} = c\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}\left((5 + 4\mu + \nu)\widehat{r_1} - 1\right). \tag{29}$$

This equation is solved for constant values of $c\tau_{\mathrm{M}}/\tau_{\mathrm{KH,e}}$ by

$$\widehat{r_1} = -\frac{1}{(5 + 4\mu + \nu)}\left(\left(\frac{M}{M_{\mathrm{A}}}\right)^k - 1\right) \tag{30}$$

where

$$k = (5 + 4\mu + \nu)c\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{KH,e}}}. \tag{31}$$

Up to first order in ϵ we find

$$R(M) = R_{\mathrm{e}}(M)\left\{1 - \frac{(\zeta_{\mathrm{e}} - \zeta_{\mathrm{ad}})}{k}\left(\left(\frac{M}{M_{\mathrm{A}}}\right)^k - 1\right)\right\} \tag{32}$$

and, by taking the derivative of (32),

$$\zeta_{\mathrm{eff}} = \zeta_{\mathrm{e}} + (\zeta_{\mathrm{ad}} - \zeta_{\mathrm{e}})\left\{\left(\frac{M}{M_{\mathrm{A}}}\right)^{-k} + \frac{(\zeta_{\mathrm{e}} - \zeta_{\mathrm{ad}})}{k}\left(\left(\frac{M}{M_{\mathrm{A}}}\right)^{-k} - 1\right)\right\}^{-1}. \tag{33}$$

We note that $\zeta_{\mathrm{eff}}(M_{\mathrm{A}}) = \zeta_{\mathrm{ad}}$ and that $\zeta_{\mathrm{eff}}(M) = \zeta_{\mathrm{e}}$ when M is sufficiently reduced such that $(M/M_{\mathrm{A}})^k \ll 1$.

Equations (32) and (33) describe the radius reaction upon mass loss of an initially undisturbed star in the vicinity of R_{e} for constant values of k and small $|\dot{M}|$.

For the case where k is not independent of M, mainly because τ_{M} varies considerably, equations (32) and (33) are still valid as long as a meaningful mean value k can be defined.

This is, in fact, possible if the duration of the time of interest is much shorter than $\tau_{\rm KH,e}$. In that case the mass losing star cannot resolve the $\dot{M}$ variation and, to a very good approximation, reacts as if mass loss proceeded on a constant mean time–scale $\overline{\tau_{\rm M}} \simeq -\overline{M}/\overline{\dot{M}}$ with $\overline{M} \simeq 0.5(M(t_{\rm A}) + M(t))$ and

$$\overline{\dot{M}} \simeq \frac{M(t_{\rm A}) - M(t)}{t_{\rm A} - t}. \tag{34}$$

4 APPLICATIONS TO CV EVOLUTION

4.1 The uniform CV evolution

The main driving mechanism of mass–transfer in CVs is thought to be angular momentum loss due to magnetic braking and gravitational radiation (e.g. King 1988). In these cases $\dot{M}$ scales as $\dot{M} \sim M^{\alpha}$ where α is a small number, i.e. $\alpha \simeq 0$ for gravitational radiation and $\alpha \simeq 0.3...1$ for magnetic braking (see e.g. Kolb & Ritter 1992). Thus in the limit of predominantly convective stars, for which $(\zeta_{\rm e} - \zeta_{\rm ad}) \simeq$ const., we derive

$$\left|\frac{\mathrm{d}\Gamma}{\mathrm{d}\ln \mathrm{M}}\right| \simeq \left|\frac{1}{c}(\beta + \alpha - 1)(\zeta_{\rm e} - \zeta_{\rm ad})\frac{\tau_{\rm KH,e}}{\tau_{\rm M}}\right| \simeq 1.5\frac{\tau_{\rm KH,e}}{\tau_{\rm M}} \ll 100\frac{\tau_{\rm KH,e}}{\tau_{\rm M}} \tag{35}$$

where $\beta = \mathrm{d}\ln\tau_{\rm KH,e}/\mathrm{d}\ln \mathrm{M} \simeq -3$.

According to equation (35) it is clear that in CV evolution in the regime of predominantly convective secondaries ($M_2 \lesssim 0.5 M_\odot$) $R_0(M)$ is a good approximation for the uniform CV evolution. We use this result in the next section.

4.2 The period gap

Near the upper edge of the period gap the donor stars are almost fully convective and CV evolution is well approximated by $R_0(M)$. Since $(\zeta_{\rm e} - \zeta_{\rm ad}) \simeq$ const. for such stars, the value of R_0 depends only on $\tau_{\rm KH,e}/\tau_{\rm M}$, i.e. on $\dot{M}$ and M. Therefore the width of the period gap $\Delta P_{\rm gap}$ for CVs which have already reached the stage of uniform CV evolution is a function only of $\tau_{\rm KH,e}/\tau_{\rm M} \sim \dot{M}$. This is shown in Fig.(3) where we plot $\Delta P_{\rm gap}/P_{\rm l}$ versus $\tau_{\rm KH,e}/\tau_{\rm M}$ taken at the point where the secondaries become fully convective. Here $P_{\rm l}$ is the period at the turn–on of mass–transfer after the detached phase. The crosses represent full stellar evolution calculations as published in Kolb (1992, 1993), Kolb & Ritter (1992), Stehle (1993) and Stehle, Kolb & Ritter (1994, 1996), the full line shows $\Delta P_{\rm gap}/P_{\rm l}$ as calculated from R_0. Setting $\mu = 0$ and $(\zeta_{\rm e} - \zeta_{\rm ad})/c = 0.44$ we obtain the best correspondence to the full stellar evolution calculations with $\nu = 6$.

4.3 The period flag

The period flag (see Ritter & Kolb 1992 for a description) which occurs at the onset of mass–transfer after the detached phase is characterized by two values of the orbital period. The first one, denoted by $P_{\rm A}$, is the period at the turn–on of mass–transfer. The other one, denoted by $P_{\rm B}$, is the period when, after the initial adiabatic expansion and subsequent thermal relaxation, $\dot{P} = 0$. The values of $\zeta_{\rm eff}$ at these two periods are respectively $\zeta_{\rm eff} = \zeta_{\rm ad} = -1/3$ when $P = P_{\rm A}$ and $\zeta_{\rm eff} = +1/3$ when $P = P_{\rm B}$ (see Fig. 4). The width of the period flag is defined by $\Delta P_{\rm fl} = P_{\rm B} - P_{\rm A}$.

The mass–transfer rate for fully convective secondaries during the period flag is typically of the order of $\sim$ few $10^{-10}\mathrm{M}_\odot/\mathrm{yr}$ so that $\tau_{\rm KH,e}/\tau_{\rm M} \simeq 0.2...0.5$ and thus $\Gamma \simeq 0.05...0.2$. Even

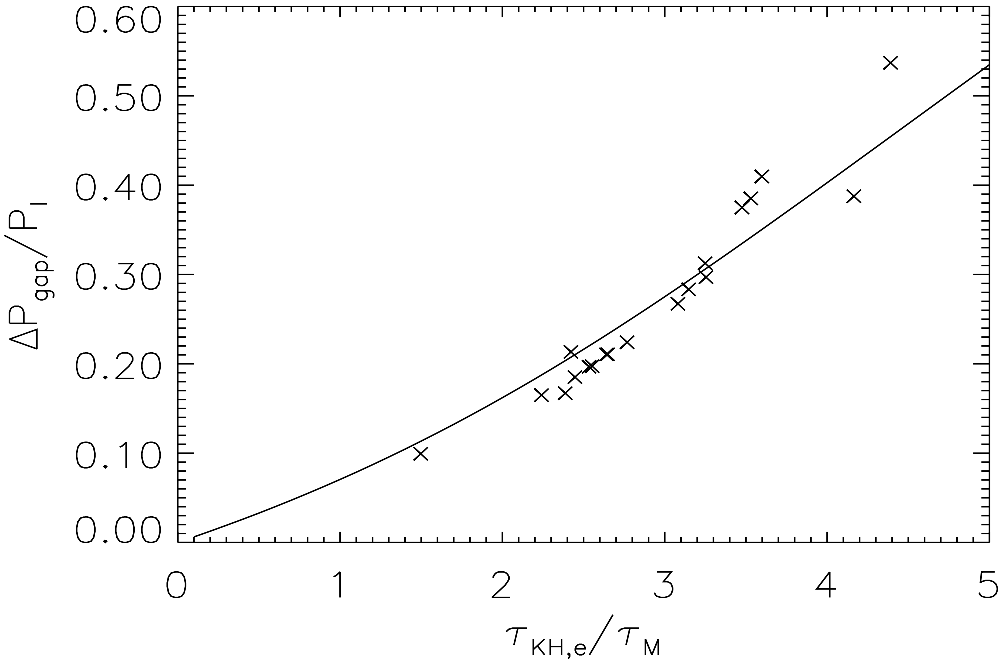

Figure 3. The relative width of the detached phase $\Delta P_{\rm gap}/P_{\rm l}$ as a function of $\tau_{\rm KH,e}/\tau_{\rm M}$ taken at the point where the secondaries become fully convective. Crosses mark full stellar evolution calculations as cited in the text. The full line shows $\Delta P_{\rm gap}/P_{\rm l}$ according to R_0 for different Γ (see text).

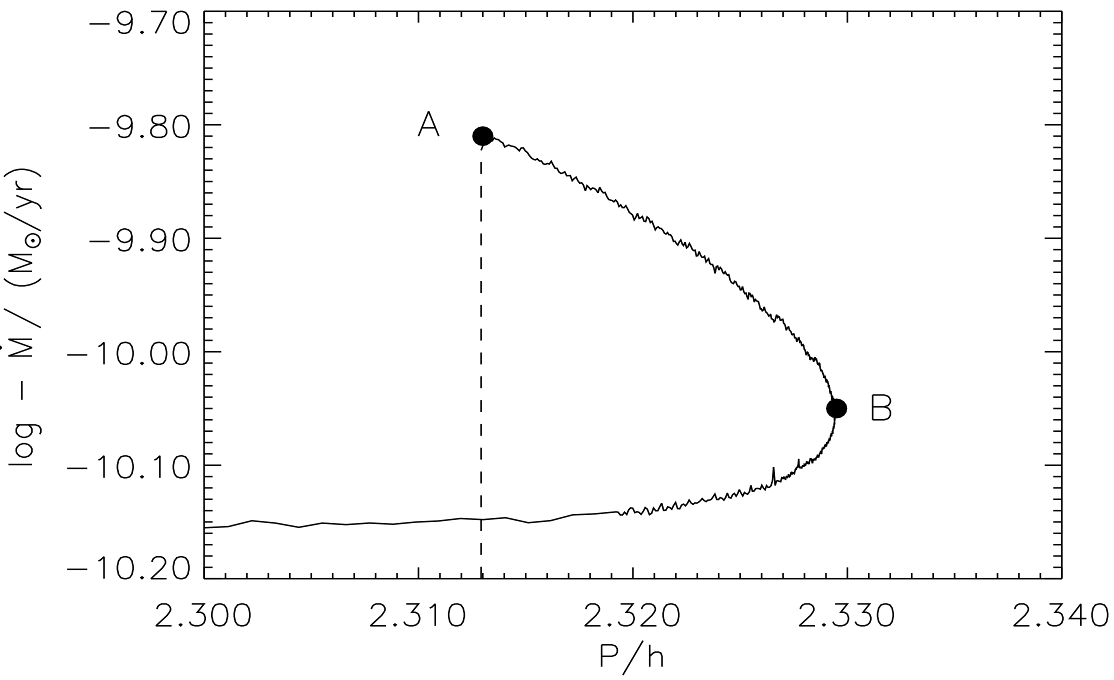

Figure 4. The period flag in CV evolution at the onset of mass–transfer after the detached phase for fully convective secondaries (Kolb 1993). The dashed line indicates the exponential increase in mass–transfer as the secondary fills its Roche lobe. From point A to point B $\zeta_{\rm eff}$ increases from $\zeta_{\rm ad}$ to 1/3.

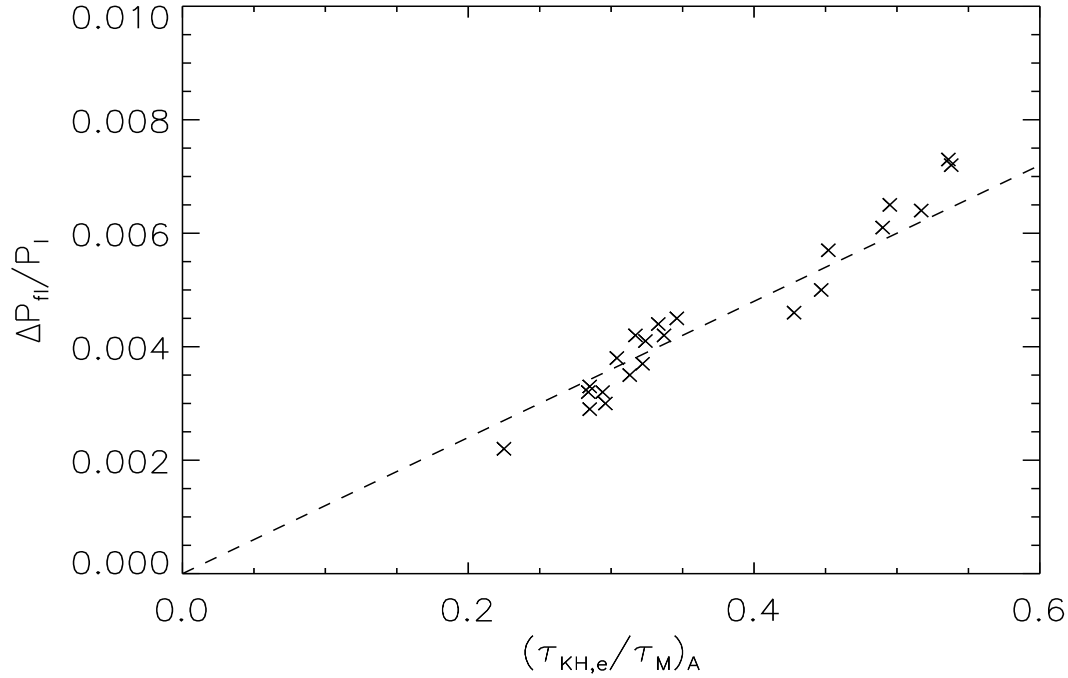

Figure 5. The relative width of the period flag $\Delta P_{\rm fl}/P_{\rm A}$ versus $\tau_{\rm KH,e}/\tau_{\rm M}$. Crosses mark full stellar evolution calculations as cited in the text, the dashed curve is a fit to the numerical values with a slope of 0.012.

so, Γ is not very much smaller than $r \simeq 1$ it is sufficiently small to allow for the perturbation expansion using Γ as a small parameter. Thus we apply the formulation of Section 3.5 to determine the width of the period flag.

We remark that the exponential increase in mass–transfer from $\dot{M} = 0$ to $\dot{M}_{\rm A}$ (dashed line in Fig. 4) where $R \neq R_{\rm R}$ is beyond the scope of our model. This neglect, however, does not influence the calculation of $\Delta P_{\rm fl}$ because the total mass–transferred in that regime is very small compared to the mass–transferred from A to B.

Using Kepler's third law and the approximation of Paczyński (1971) for the Roche–radius, we obtain, by expanding (**??**) up to first order in k^{-1},

$$\frac{\Delta P_{\rm fl}}{P_{\rm A}} \simeq \ln\left(\frac{P_{\rm B}}{P_{\rm A}}\right) = \frac{3}{2}\ln\left(\frac{R_{\rm B}}{R_{\rm A}}\right) - \frac{1}{2}\ln\left(\frac{M_{\rm B}}{M_{\rm A}}\right). \tag{36}$$

Because $\zeta_{\rm eff}(M_{\rm B}) = +1/3$, we obtain by expanding (33) up to first order in k^{-1}

$$\ln\frac{M_{\rm B}}{M_{\rm A}} = -\frac{2}{3\,\zeta_{\rm e}\,k} = -\frac{2}{3\,\zeta_{\rm e}\,c(5+4\mu+\nu)}\frac{\tau_{\rm KH,e}}{\tau_{\rm M}}. \tag{37}$$

On the other hand, in the case of constant $\tau_{\rm M}$, we have

$$\ln\frac{M_{\rm B}}{M_{\rm A}} = \int_{t_{\rm A}}^{t_{\rm B}} \frac{\rm d\ln M}{\rm dt}{\rm dt} = -\int_{\rm t_A}^{\rm t_B}\frac{\rm dt}{\tau_{\rm M}} = -\frac{\rm t_{AB}}{\tau_{\rm M}}, \tag{38}$$

whereas in the case of variable $\tau_{\rm M}$ (and $M_{\rm A} - M_{\rm B} \ll M_{\rm A}$)

$$\ln\frac{M_{\rm B}}{M_{\rm A}} = \int_{t_{\rm A}}^{t_{\rm B}}\frac{\dot{M}}{M}{\rm dt} \simeq \frac{1}{\overline{\overline{\rm M}}}\int_{\rm t_A}^{\rm t_B}\dot{\rm M}\,{\rm dt} \simeq \frac{\overline{\dot{\rm M}}}{\overline{\overline{\rm M}}}{\rm t_{AB}} = -\frac{\rm t_{AB}}{\overline{\tau_{\rm M}}}. \tag{39}$$

Here $t_{\rm AB} = t_{\rm B} - t_{\rm A}$ is the time it takes the system to evolve from point A to point B, where respectively $P = P_{\rm A}$, $M = M_{\rm A}$ and $P = P_{\rm B}$, $M = M_{\rm B}$. As we have argued above, in the case of variable mass–transfer rate, $\tau_{\rm M}$ in (37) can be replaced by $\overline{\tau_{\rm M}}$ defined by (39) if $t_{\rm AB} \ll \tau_{\rm KH,e}$. Combining now (37) and (39) we find that

$$t_{\rm AB} = \frac{2}{3}\frac{1}{\zeta_{\rm e} c(5+4\mu+\nu)}\tau_{\rm KH,e} = \frac{2}{3\zeta_{\rm e}}\tau_{\rm per} \tag{40}$$

is indeed much shorter than $\tau_{\rm KH,e}$ (typically about $1/30\,\tau_{\rm KH,e}$), and independent of $\tau_{\rm M}$. Because $t_{\rm AB} \ll \tau_{\rm KH,e}$, this result holds also for the case of varying $\tau_{\rm M}$. With (40) we have derived analytically a result which is known since long from numerical computations (see, e.g., Ritter & Kolb 1992, Table 1), namely that in a first approximation $t_{\rm AB}$ is independent of $\tau_{\rm M}$. In order to evaluate (36) we still need $\ln R_{\rm B}/R_{\rm A}$. This is obtained from (32) by eliminating $(M/M_{\rm A})$ by means of (33). Accordingly,

$$\ln\frac{R_{\rm B}}{R_{\rm A}} \simeq \frac{2}{3k} + \zeta_{\rm e}\ln\frac{M_{\rm B}}{M_{\rm A}}. \tag{41}$$

This, together with (36), (37) and (40), finally yields

$$\frac{\Delta P_{\rm fl}}{P_{\rm A}} \simeq \frac{1}{2}t_{\rm AB}\frac{\overline{\dot M}}{\overline{M}} = \frac{1}{3k\zeta_{\rm e}} \simeq \frac{1}{3\zeta_{\rm e} c(5+4\mu+\nu)}\left(\frac{\tau_{\rm KH,e}}{\overline{\tau_{\rm M}}}\right). \tag{42}$$

This result shows that $\Delta P_{\rm fl}$ is proportional to the mass $\delta M = \overline{\dot M}\, t_{\rm AB}$ which is lost during the evolution from A to B. Furthermore, (42) predicts that $\Delta P_{\rm fl}/P_{\rm A}$ is a linear function of $\tau_{\rm KH,e}/\tau_{\rm M}$. This prediction can be checked against results of numerical calculations. This is done in Fig.5, where we plot results obtained by Kolb (1992, 1993), Kolb & Ritter (1992), Stehle (1993) and Stehle et al. (1994, 1996) as crosses as a function of $\tau_{\rm KH,e}/\tau_{\rm M}$ taken at point A. The dashed line is a linear fit to the numerical results and has a slope of 0.012. In order to make a meaningful comparison with equation (42), we have to replace $\overline{\tau_{\rm M}}$ in (42) by $\tau_{\rm M}$ at point A. From numerical calculations (e.g. D'Antona et al. 1989) we know that between points A and B $\ln(-\dot M)$ decreases roughly linearly with time. Thus for typical values of the mass–radius exponent of the Roche radius $\zeta_{\rm R}$

$$\overline{\dot M} \simeq \sqrt{\dot M_{\rm A}\dot M_{\rm B}} = \dot M_{\rm A}\left(\frac{\dot M_{\rm B}}{\dot M_{\rm A}}\right)^{1/2} = \dot M_{\rm A}\left(\frac{-1/3-\zeta_{\rm R}}{1/3-\zeta_{\rm R}}\right)^{1/2} \simeq 0.8\dot M_{\rm A}. \tag{43}$$

Therefore, with $c = 2.6, \mu = 0$, $\zeta_{\rm e} = 0.8$ and $\nu = 6$, the predicted slope is

$$\frac{0.8}{3\zeta_{\rm e} c(5+4\mu+\nu)} \simeq 0.012, \tag{44}$$

in perfect agreement with the numerical results.

5 NUMERICAL INTEGRATION

5.1 Thermal equilibrium models

For a numerical integration of equation (9) we need to know the properties of thermally relaxed stars. For this we have calculated a sufficiently dense grid of ZAMS models of PopI ($Z = 0.02, Y = 0.25$) and of PopII ($Z = 10^{-4}$, $Y = 0.25$) chemical composition, using the full stellar evolution code of Mazzitelli (1989). With these data we derived the relations $R_{\rm e}(M)$ and $\tau_{\rm KH,e}(M)$ by spline interpolation and $\zeta_{\rm e}(M)$ by numerical differentiation of $R_{\rm e}(M)$. For $\zeta_{\rm ad}$ we interpolated results obtained by Hjellming (1989) as a function of the relative mass of the convective envelope. In Fig. 6 we show the resulting values of $\zeta_{\rm ad}$ and $\zeta_{\rm e}$ for the PopI and PopII models as a function of mass.

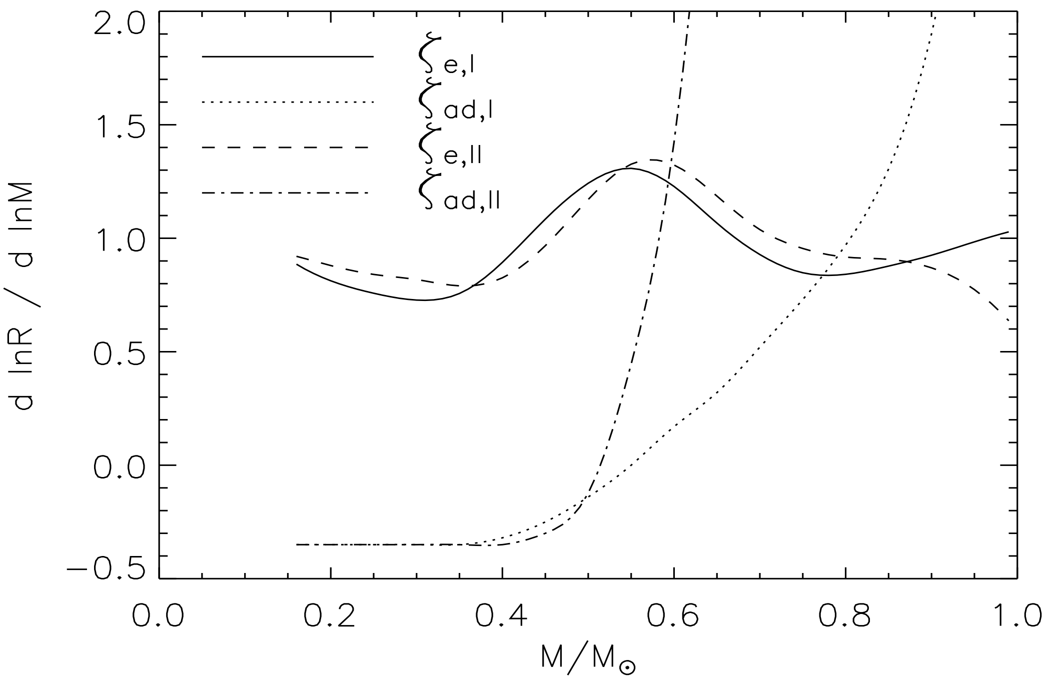

Figure 6. The mass–radius exponent ζ_e for thermally relaxed stars and the adiabatic mass–radius exponent ζ_{ad} both for PopI and PopII stars.

5.2 Evolution with constant $(\tau_{KH,e}/\tau_M)$

First, we integrate equation (9) with a constant ratio $\tau_{KH,e}/\tau_M$. As an example, we show, in Fig. 7 R(M) computed for $\tau_{KH,e}/\tau_M = 2$ as a dotted line, and for comparison $R_0(M)$ (dashed) and $R_e(M)$ (full line) for PopI (Fig. 7a) and PopII (Fig. 7b) chemical composition. As can be seen, the results show the expected behaviour: In response to mass loss a ZAMS star which is initially mainly radiative (thus $\zeta_e(M_i) < \zeta_{ad}(M_i)$) contracts with respect to the ZAMS. In the case of PopII composition and $M \gtrsim 0.7 M_\odot$, the companion contracts strongly due to its high value of ζ_{ad}. Therefore (cf. Section 3.4.) the evolution after a time–scale τ_{per}, is well approximated by $R_0(M)$. However, if $\zeta_e \simeq \zeta_{ad}$ (i.e. $M \simeq 0.4...0.7 M_\odot$ for PopII composition, $M \simeq 0.6...0.9 M_\odot$ for PopI composition), R_0 is not as good an approximation to the CV evolution.

In Fig. 8 we show integrations for different values of $\tau_{KH,e}/\tau_M$. It is remarkable that tracks nearly coincide during the initial contraction phase. This is particularly well seen in the case of PopII composition (Fig. 8b). Furthermore, it is seen that they cross the ZAMS line nearly at the same mass.

5.3 An example of a full secular evolution

Next, we wish to compare the results of a full secular evolution of a CV with results obtained by integrating equation (9). The evolutionary calculations have been done with Mazzitelli's (1989) stellar evolution code, assuming PopII chemical composition, an initial secondary mass $M_{2,i} = 0.9 M_\odot$, a white dwarf mass $M_1 = 1 M_\odot$ (kept constant during the evolution) and angular momentum loss due to magnetic braking (Verbunt & Zwaan 1981, $f_{VZ} = 1$) and gravitational radiation (e.g. Landau & Lifshitz 1975). For details see Stehle et al. (1996). The result of this calculation is shown in Fig. 9 as a full line. As a dash–dotted line we

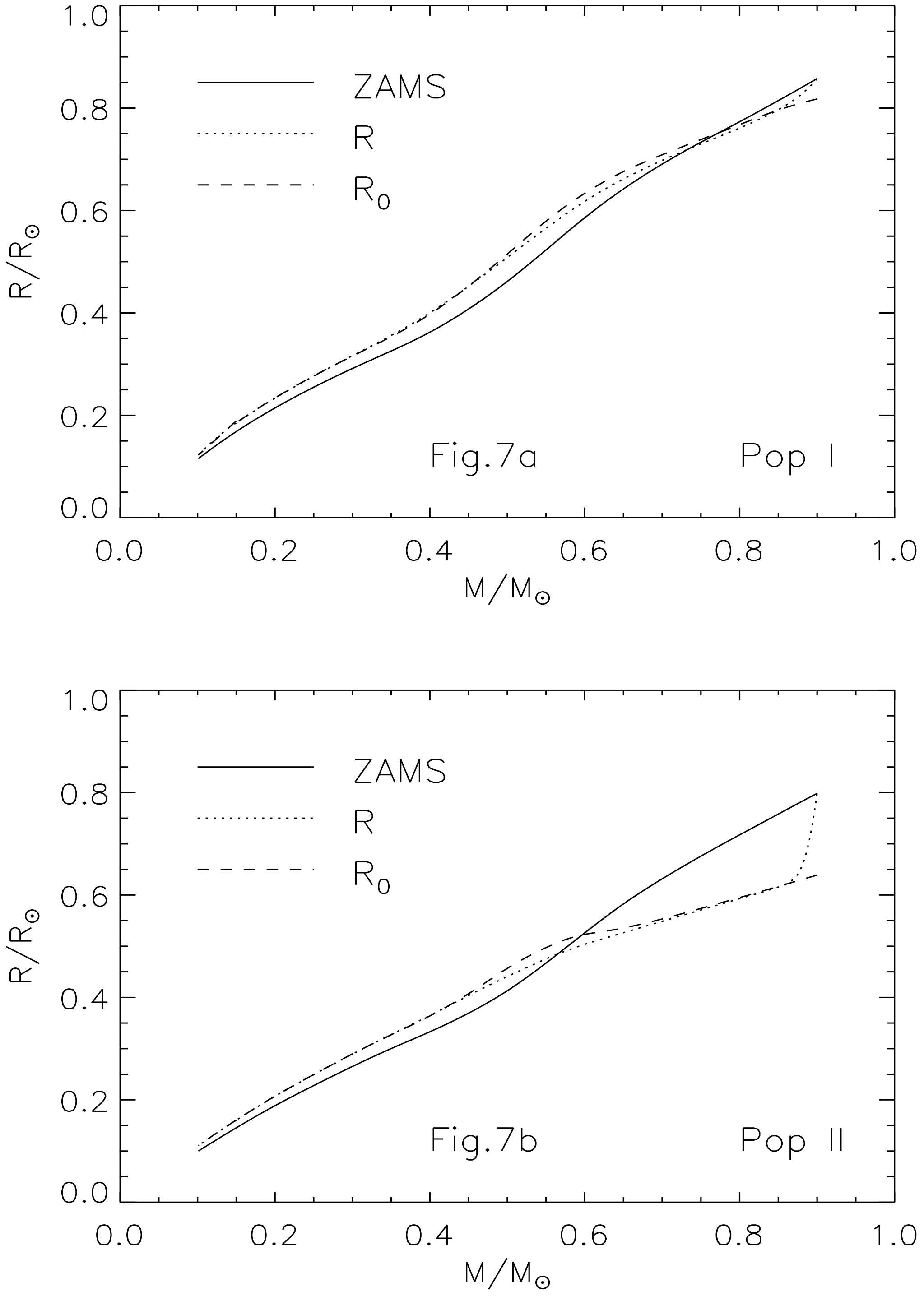

Figure 7. The stellar radius R derived from the numerical integration of equation (9) with $\tau_{\rm KH,e}/\tau_{\rm M} = 2$ (dotted curve) and its approximation $R_0(M)$ (dashed line) by equation (15), is plotted versus the stellar mass M. The thermal equilibrium radius $R_{\rm e}(M)$ is shown as a full line, for both PopI (a) and PopII (b) chemical composition.

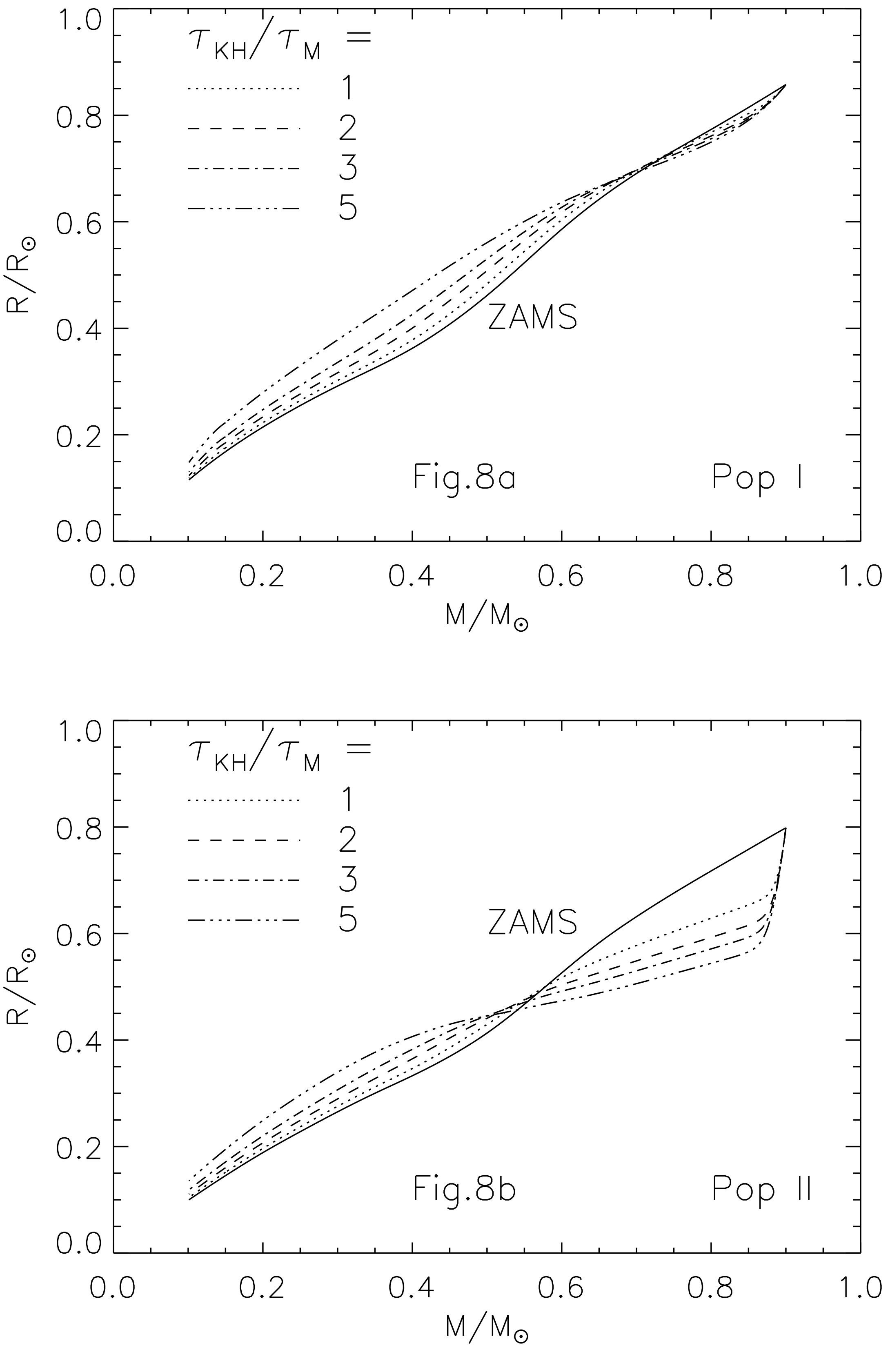

Figure 8. As Figs. 7a and 7b but for different values of $\tau_{\mathrm{KH,e}}/\tau_{\mathrm{M}}$.

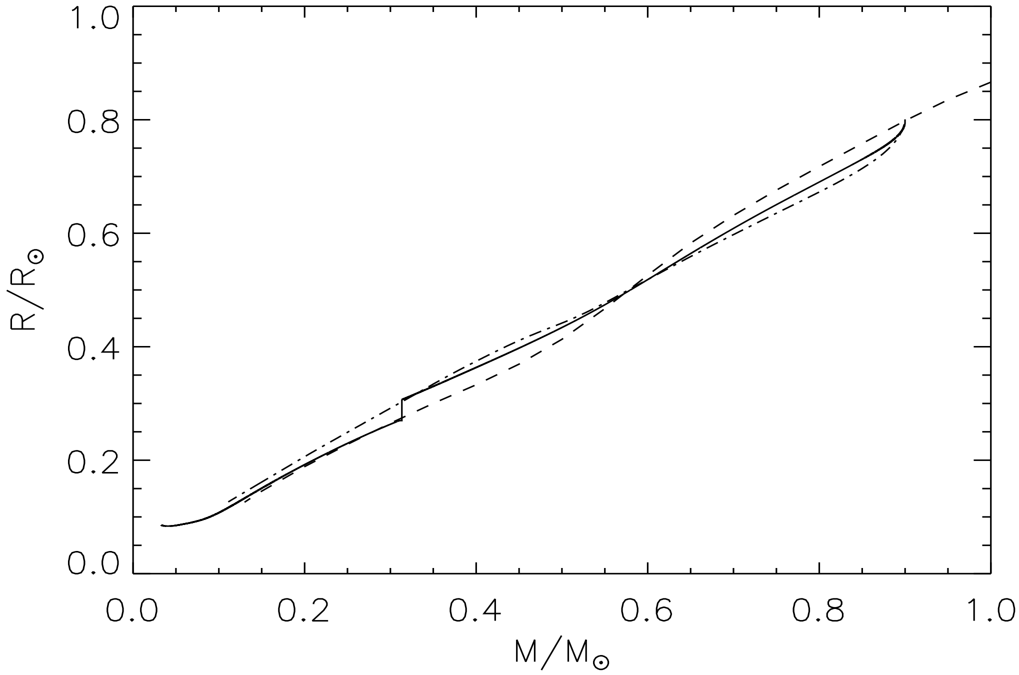

Figure 9. A comparison of a full stellar evolution calculation of a PopII CV (full line) and the integration of equation (9) (dash–dotted curve) using in both cases the Verbunt & Zwaan (1981) formulation for magnetic braking as the source of angular momentum loss. In addition, we plotted as a dashed curve the ZAMS mass–radius relation.

show the integration of equation (9) together with equation (12) assuming the same orbital angular momentum loss as in the full stellar evolution calculation. In addition the ZAMS mass–radius relation is plotted as a dashed line. As can be seen, where the comparison applies, i.e. in the mass range $M_{\rm conv} < M_2 < M_{2,\rm i}$ (where $M_{\rm conv}$ denotes the mass at which the star becomes fully convective), the agreement of the full evolutionary calculation and the corresponding solution of equation (9) with $\nu = 10$, $\mu = 0$ and $a = 0.5$ is quite good. By this we mean that the overall features of the evolution are well reproduced. The most surprising aspect of this is, however, that the good agreement also holds for secondaries in the mass range $M_2 > 0.6 M_\odot$, where the relative mass of the convective envelope is rather small, and for which our simple model is, strictly speaking, not applicable.

Finally, we should stress that despite the overall good agreement between the results of full stellar evolution calculations and the solution of equation (9), solving only the latter cannot substitute for the full calculations. The main reasons for this are that, on the one hand, equation (9) is derived from a rather simple stellar model which, however, does not apply to the full range of interest (e.g., for stars with $M > 1 M_\odot$ or if the star in question becomes degenerate) and, on the other hand, the model involves a number of parameters such as ν, μ or a which have to be gauged anyway with full stellar calculations if equation (9) is to reproduce full calculations quantitatively.

6 SUMMARY AND CONCLUSIONS

We have studied the reaction of a low–mass star upon mass loss by using simple homology relations to describe stellar structure. In the framework of this model we show that the reaction of the stellar radius R to mass loss is described by a first–order differential equation.

This differential equation becomes particularly simple if the change of R/R_e, where R_e is the corresponding thermal equilibrium radius, is considered. The resulting equation (9) then contains coefficients which depend only on the properties of the star in thermal equilibrium and on the mass loss rate. Thus, within the range of applicability of our model, the properties of a mass losing star are always strongly influenced by the properties of the corresponding thermal equilibrium model.

The main properties of this differential equation are as follows.

(i) All evolutions with different initial conditions tend to converge to a single uniform evolution.

(ii) The evolution towards this uniform solution is characterized by a time–scale $\tau_{\rm per}$ which is short compared to the star's Kelvin–Helmholtz time in thermal equilibrium, $\tau_{\rm KH,e}$. Typically we find $\tau_{\rm per} \lesssim 0.05\tau_{\rm KH,e}$. Within a few times $\tau_{\rm per}$ a star forgets about the initial conditions of its evolution with mass loss and follows subsequently the uniform evolution.

(iii) The shape of the single uniform evolution, say in the mass–radius diagram, depends only on the mass–radius relation in thermal equilibrium and on the time–scale of mass loss averaged over the past few $\tau_{\rm per}$. Since $\tau_{\rm per}$ is also a measure for a star's thermal inertia, $\dot{M}$–fluctuations with frequencies higher than $\tau_{\rm per}^{-1}$ have little or no effect on the stellar radius.

(iv) If the time–scale $|\dot{M}/\ddot{M}|$ on which $\dot{M}$ changes is long compared to $\tau_{\rm per}$, the uniform evolution is essentially given by the momentary value of $\dot{M}$.

(v) If these two time–scales are comparable the secular evolution can only be determined by integrating equation (9) explicitly.

The existence of the uniform radius evolution of a mass losing star has the following important consequences for the secular evolution of CVs:

(1) Since $\tau_{\rm per}$ is so short, evolutionary tracks of CVs which differ only in the initial mass of the secondary converge rapidly to a uniform evolutionary track. This behaviour, which was first noticed by Paczyński & Sienkiewicz (1983), is a direct consequence of the existence of the above–described attracting solution.

(2) If the angular momentum loss rate in CVs above the period gap, i.e. with $P > 3$h, is sufficiently insensitive to the mass of the white dwarf, the short convergence time–scale $\tau_{\rm per}$ guarantees that the majority of CVs enter the detached phase with secondaries which have essentially the same mass and the same radius excess, i.e. the same R/R_e. This, in turn, provides the coherence which is necessary to allow the detached phase of a single CV evolution to appear as the period gap of the CV population.

(3) CVs from a subpopulation which is characterized by a given mass of the white dwarf (or a sufficiently small mass interval) and which enter the detached phase with the secondary being near to the uniform evolution turn on mass–transfer below the gap at essentially the same orbital period. This is one of the prerequisites necessary for explaining the observed period spike of AM Her systems at $P \simeq 114$min (see e.g. Hameury et al. 1988, Ritter & Kolb 1992). By linearizing equation (9), we have furthermore shown that the width of the period flag at turn–on of mass–transfer below the gap $\Delta P_{\rm fl}/P_{\rm A} \sim \dot{M}$ and that for the canonical mechanism of the angular momentum loss (gravitational radiation) $\Delta P_{\rm fl}/P_{\rm A} \ll 1$. This is consistent with the explanation of the period spike as the turn–on phase of mass–transfer after the detached phase.

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