Feynman's sunshine numbers

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Abstract: This is an expansion of a talk for mathematics and physics students of the Manchester Grammar and Manchester High Schools. It deals with numbers such as the Riemann zeta value $\zeta(3) = \sum_{n>0} 1/n^3$. Zeta values appear in the description of sunshine and of relics from the Big Bang. They also result from Feynman diagrams, which occur in the quantum field theory of fundamental particles such as photons, electrons and positrons. My talk included 7 reasonably simple problems, for which I here add solutions, with further details of their context.

1 Numbers

1.1 Sum of the first 136 cubes

$$\begin{aligned} 1^3 &= 1 = 1^2 \\ 1^3+2^3 &= 9 = (1+2)^2 \\ 1^3+2^3+3^3 &= 36 = (1+2+3)^2 \\ 1^3+2^3+3^3+\ldots+136^3 &\stackrel{?}{=} (68\times 137)^2 = (1+2+3+\ldots+136)^2 \end{aligned}$$

It is rather easy to prove, by induction, that

$$\sum_{n=1}^{N-1} n^3 = \frac{1}{4}N^2(N-1)^2 \qquad (1)$$

for every integer $N \geq 2$.

Assume that formula (1) is true for one particular integer, say $N = M$. Then

$$\sum_{n=1}^{M} n^3 = \frac{1}{4}M^2(M-1)^2 + M^3 = \frac{1}{4}M^2(M+1)^2$$

and (1) is true for $N = M + 1$. It is true, by inspection, for $N = 2$, and hence for every integer $N > 1$.

1.1.1 Problem 1: proof by induction

Prove that, for every integer $N > 1$,

$$\sum_{n=1}^{N-1} n^4 = \frac{1}{30}N(N-1)(2N-1)(3N^2-3N-1). \qquad (2)$$

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1.2 Sum of the first 136 inverse cubes

$$
\begin{aligned}
\sum_{n=1}^{2} \frac{1}{n^3} &= 1 + \frac{1}{8} = \frac{3^2}{2^3} \\
\sum_{n=1}^{3} \frac{1}{n^3} &= 1 + \frac{1}{8} + \frac{1}{27} = \frac{251}{2^3 \times 3^3} \\
\sum_{n=1}^{4} \frac{1}{n^3} &= \frac{5 \times 11 \times 37}{2^6 \times 3^3} \\
\sum_{n=1}^{6} \frac{1}{n^3} &= \frac{7^2 \times 11 \times 53}{2^6 \times 3 \times 5^3} \\
\sum_{n=1}^{8} \frac{1}{n^3} &= \frac{31 \times 2538983}{2^9 \times 3 \times 5^3 \times 7^3} \\
\sum_{n=1}^{10} \frac{1}{n^3} &= \frac{11^2 \times 89 \times 359 \times 4957}{2^9 \times 3^6 \times 5^3 \times 7^3} \\
\sum_{n=1}^{12} \frac{1}{n^3} &= \frac{13^2 \times 151099201553}{2^9 \times 3^6 \times 5^3 \times 7^3 \times 11^3}
\end{aligned}
$$

It is quite hard to obtain the complete factorization of the numerator of the sum of the first 136 inverse cubes. Let

$$S(N) = \sum_{n=1}^{N-1} \frac{1}{n^3}$$

then the numerator of $S(137)$ turns out to be

$$137^2 \times 359 \times 20273371 \times 20077753681 \times 95441340948666564707 \times P_{29} \times P_{47} \times P_{54}$$

where

$$
\begin{aligned}
P_{29} &= 11670683543184939914296499797 \\
P_{47} &= 15515586948180802047607109239107700793654018233 \\
P_{54} &= 135756874001680563462725618538328344771913356122240503
\end{aligned}
$$

are primes with 29, 47 and 54 decimal digits.

1.2.1 Problem 2: modular arithmetic

Find all the primes p such that p^2 divides the numerator of $S(p)$, using the result that for every prime p and every integer n with $p > n > 0$ there is a unique integer m with $p > m > 0$ and $mn - 1$ divisible by p. [Hints follow.]

1.2.2 Hints for Problem 2

Let p be any prime. Let a and b be rational numbers with denominators that are not divisible by p. If the numerator of $a-b$ is divisible by p, we say that a and b are congruent modulo p. Let's agree to abbreviate this by writing $a \equiv b \pmod{p}$. Begin by showing that

$$2S(p) = \sum_{n=1}^{p-1} \left(\frac{1}{n^3} + \frac{1}{(p-n)^3} \right) \equiv 0 \pmod{p}.$$

Now show that $2S(p)/p \equiv -3T(p) \pmod{p}$, where

$$T(p) = \sum_{n=1}^{p-1} \frac{1}{n^4}.$$

We are told that for every integer n with $p > n > 0$ there is a unique integer m with $p > m > 0$ and $m \equiv 1/n \pmod{p}$. Hence you may now use the result of Problem 1 to solve Problem 2.

1.3 Riemann zeta values

Now let's consider what happens as $N \to \infty$ in the sum $S(N) = \sum_{N>n>0} 1/n^3$. The answer has a name: $\zeta(3)$, where $\zeta(s)$ is the value of the Riemann zeta function with argument s. Bernhard Riemann (1826–1866) found remarkable relations between the zeros of $\zeta(s)$ in the complex plane and the distribution of prime numbers. Here, we shall study only the Riemann zeta values

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for integers $s > 1$.

`Pari-GP` is a computer algebra system designed for fast computations in number theory and is available from `http://pari.math.u-bordeaux.fr` for free. It works well on `Windows` and `linux` platforms. Here, I ask `Pari-GP` to print values of $\pi^s/\zeta(s)$ for 5 even integers $s \geq 2$, at 60-digit precision.

```
default(realprecision, 60);
forstep(s=2,10,2, print([s, Pi^s/zeta(s)]));
```

The output

```
[2, 6.00000000000000000000000000000000000000000000000000000000000]
[4, 90.0000000000000000000000000000000000000000000000000000000000]
[6, 945.000000000000000000000000000000000000000000000000000000000]
[8, 9450.00000000000000000000000000000000000000000000000000000000]
[10, 93555.0000000000000000000000000000000000000000000000000000000]
```

strongly suggests (but does not prove) the exact evaluations

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(4) &= \frac{\pi^4}{90} \\
\zeta(6) &= \frac{\pi^6}{945} \\
\zeta(8) &= \frac{\pi^8}{9450} \\
\zeta(10) &= \frac{\pi^{10}}{93555}
\end{aligned}$$

which were proven in 1735 by the master analyst Leonhard Euler (1707–1783). For every *even* integer $s \geq 2$, it is proven that $\zeta(s)/\pi^s$ is a rational number.

For folk who love analysis (and only for them) here is a cunning method to prove that $\zeta(2) = \pi^2/6$. First one may use binomial expansions to show that

$$\zeta(2) = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{1 - xy} \tag{3}$$

$$\frac{1}{2}\zeta(2) = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{1 + xy}. \tag{4}$$

Adding these results and dividing by 2, we obtain

$$\frac{3}{4}\zeta(2) = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{1 - x^2 y^2}. \tag{5}$$

Then the delicate transformations

$$x = \frac{\sin(a)}{\cos(b)} \tag{6}$$

$$y = \frac{\sin(b)}{\cos(a)} \tag{7}$$

give

$$\frac{3}{4}\zeta(2) = \int_0^{\frac{\pi}{2}} \mathrm{d}a \int_0^{\frac{\pi}{2} - a} \mathrm{d}b = \frac{\pi^2}{8} \tag{8}$$

and hence $\zeta(2) = \pi^2/6$.

1.3.1 Problem 3: evaluating $\zeta(2)$

Prove equations (3), (4), (5) and (8). [This is for enthusiasts of analysis, for whom hints follow. An apocryphal remark by Sherlock Holmes appears in the solution.]

1.3.2 Hints for Problem 3

Show that the integrals in (3) and (4) give $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$ and $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\ldots$. Hence prove (4) by considering the difference of these sums. Then work out the Jacobian of the transformations (6) and (7), namely the determinant

$$J(x,y) = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b \\ \partial y/\partial a & \partial y/\partial b \end{vmatrix}, \tag{9}$$

and obtain the result in (8) from (5).

1.4 Is $\zeta(3)/\pi^3$ an irrational number?

It is proven that $\zeta(3)$ is not a rational number, but it is not known, for certain, whether $\zeta(3)/\pi^3$ is irrational. It seems, however, overwhelmingly probable that this number is irrational. There are rather good algorithms with which to guess integers N and D such that $\zeta(s)/\pi^s$ is close to N/D. For *odd* integers $s \geq 3$, no such attempt has produced a guess that works. For even integers $s \geq 2$, this experimental method works rather well.

1.4.1 Problem 4: the PSLQ and LLL algorithms

Use the `lindep` procedure of `Pari-GP`, at 100-digit precision, to search for an integer relation between the three constants in the vector $[\zeta(12),\, \pi^{12},\, 1]$. Check that the PSLQ and LLL options of `lindep` give the same result and that the resulting formula for $\zeta(12)$ still holds at 1000-digit precision. Finally, investigate, with comparable vigour, whether there might be an integer relation between the constants in the vector $[\zeta(13),\, \pi^{13},\, 1]$. [This problem is for enthusiasts of free-ware, for whom hints follow.]

1.4.2 Hints for Problem 4

In the problem, you are asked to perform investigations for $\zeta(12)$ and $\zeta(13)$. Here, I indicate how I used `Pari-GP` to investigate $\zeta(2)$ and $\zeta(3)$. I began with

```
default(realprecision, 100); flag = -3;
print(lindep([zeta(2), Pi^2, 1], flag)~);
```

which gave the answer $[-$`6, 1, 0`$]$. This means that, at 100-digit precision, the procedure `lindep` "guessed" the integer relation $-6 \times \zeta(2) + 1 \times \pi^2 + 0 \times 1 = 0$. The flag -3 invokes the PSLQ algorithm of Helaman Ferguson and David Bailey. Note that I introduced the "red-herring" constant 1 as a safety measure. Next, I changed the flag to 0, so as to invoke the LLL algorithm of Arjen Lenstra, Hendrik Lenstra and László Lovász, obtaining the same integer relation. To be even safer, I evaluated $6\zeta(2) - \pi^2$ at 1000-digit precision, obtaining the answer `0.E-1001` . Then, I investigated the vector $[\zeta(3),\ \pi^3,\ 1]$, obtaining no credible integer relation from either PSLQ or LLL. The PSLQ algorithm sensibly refused to give me a relation, returning instead a large real constant as an error message. The LLL algorithm gave me a relation, but this was soon shown to be untenable by increasing the precision.

2 Sunshine (and Big Bang) numbers

2.1 The Stefan–Boltzmann constant

A black body is one that absorbs electromagnetic radiation with maximum efficiency and emits it with maximum efficiency, at all frequencies f. It appears black only when very cold. When it gets hot it will glow, first red-hot and then at higher temperatures T white-hot. The spectrum of its radiation depends on f only via the convenient dimensionless ratio

$$x = \frac{hf}{kT}$$

where h is Planck's constant and k is Boltzmann's constant. If the temperature, T, of the black body (relative to absolute zero) is given in Kelvins (K) and the frequency, f, is given in Herz (Hz), then the approximate values $h = 6.626 \times 10^{-34}$ J s and $k = 1.381 \times 10^{-23}$ J K^{-1} will serve here.

The spectrum of sunlight that we receive at the top of the Earth's atmosphere is rather similar to the spectrum produced by a black body with $T \approx 6000$ K. The much spikier spectrum that we receive at sea level results from the complex composition of our atmosphere and may be subject to dramatic change, in your lifetime, by our carbon emissions.

Electromagnetic energy is emitted by a black body of surface area A and temperature T at rate given by $\sigma A T^4$, where

$$\sigma = \frac{2\pi^5}{15}\,\frac{k^4}{h^3c^2} = 5.670 \times 10^{-8}\ \text{J m}^{-2}\ \text{s}^{-1}\ \text{K}^{-4} \tag{10}$$

is called the Stefan–Boltzmann constant. It has a dependence on h, k and the speed of light $c = 2.998 \times 10^8$ m s^{-1} that is easily found by dimensional analysis. (Just look at the units of σ and you can do it in your head.)

But where, on Earth, does that Stefan–Boltzmann factor

$$\frac{2\pi^5}{15} \approx 40.80$$

come from, for goodness sakes?

Amusingly, it turns out that we don't need the precise value of this constant to relate the temperature of the Earth, T_{E}, to the temperature of the surface of the Sun, T_{S}. If we assume that both the Earth and the Sun radiate like black bodies, then equilibrium, on Earth, is achieved when

$$\sigma(4\pi R_{\rm E}^2)T_{\rm E}^4 \approx \left(\frac{R_{\rm E}}{2D}\right)^2 \sigma(4\pi R_{\rm S}^2)T_{\rm S}^4$$

where $R_{\rm E}$ and $R_{\rm S}$ are the radii of the Earth and Sun and D is the distance between them. Note that the first factor on the right hand side is the fraction of the Sun's power that reaches us and is determined simply by geometry. Thus we don't need the values of σ or $R_{\rm E}$ and arrive at the estimate

$$\frac{T_{\rm E}}{T_{\rm S}} \approx \frac{1}{2}\sqrt{\frac{2R_{\rm S}}{D}} \approx \frac{1}{2}\sqrt{\frac{32}{60}\frac{2\pi}{360}} \approx 0.048$$

using only the average angular diameter of the Sun, which is about 32 minutes of arc. If we take $T_{\rm E} = 288$ K (about 15 degrees Celsius) as the average temperature of the Earth, then we estimate that $T_{\rm S} \approx T_{\rm E}/0.048 = 6000$ K.

2.2 Planck's integral

In 1900, Max Planck (1858–1947) gave a formula for the black-body spectrum, i.e. the probability that the energy emitted by a black body lies in the narrow range of frequencies between f and $f+{\rm d}f$. This is proportional to $B(x){\rm d}x$, with $x = hf/(kT)$ and

$$B(x) = \frac{2\pi x^3}{\exp(x) - 1}.$$

When I read about this, in 1962, it was clear that there was an interesting integral to do. To derive the factor of 40.80 in the Stefan-Boltzmann constant, one needs to prove that

$$\int_0^\infty \frac{2\pi x^3\,{\rm d}x}{\exp(x) - 1} = \frac{2\pi^5}{15}$$

but the school library only seemed to contain texts saying things like "it can be shown that…" this evaluation is correct. There was no internet to consult. Nor was there a convenient servant like this

```
default(realprecision, 60);
lhs = intnum(x=0,[[1],1], 2*Pi*x^3/(exp(x)-1)); print(lhs);
40.8026246380375271016988413391247475307067608837433320737990
rhs = 2*Pi^5/15; print(rhs);
40.8026246380375271016988413391247475307067608837433320737990
```

available to a 15-year-old in the early 1960's, to lend such numerical reassurance.

2.2.1 Problem 5: the sunshine number $\zeta(4)$

By expanding in powers of $\exp(-x)$ show that

$$\int_0^\infty \frac{2\pi x^3\,{\rm d}x}{\exp(x) - 1} = 12\pi\zeta(4).$$

2.2.2 Hints for Problem 5

You will need to prove that

$$\int_0^\infty x^3 \exp(-nx)\,\mathrm{d}x = \frac{3!}{n^4}.$$

To do so, use the integration variable $y = nx$ and integrate by parts, three times.

2.3 Evaluations of $\zeta(4)$

Thus the problem of the power of sunshine has been pushed back to Euler's 1735 result that $\zeta(4) = \pi^4/90$.

2.3.1 $\zeta(4)$ from the cotangent function

It is a deep result in complex analysis that we have the wonderful formula

$$\frac{\cos(z)}{\sin(z)} = \sum_{n=-\infty}^{\infty} \frac{1}{z - n\pi}. \tag{11}$$

The right hand side has the same singularities, at $z = n\pi$, with the same unit residues as occur for the cotangent, on the left hand side. Thus the difference between the right and left hand sides is an entire function, with no singularities in the complex plane. That does not, of itself, mean that this difference is the zero function, since $\exp(z)$ is an example of an entire function that does not vanish. Let's take (11) on trust, from deeper thinkers, and use it to show that $\zeta(4) = \pi^4/90$. First we should multiply by z, to remove the singularity at $z = 0$, and then combine the terms with positive and negative n, to obtain

$$\frac{z\cos(z)}{\sin(z)} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}$$

with a sum that converges near $z = 0$. We know how to expand the left hand side, as $z \to 0$, using a ratio of very simple Taylor series for $\cos(z)$ and $\sin(z)$. On the right hand side we find the constants $\zeta(2k)/\pi^{2k}$, after binomial expansion. In particular, by simplifying the relation

$$\frac{1 - z^2/2! + z^4/4! + O(z^6)}{1 - z^2/3! + z^4/5! + O(z^6)} = 1 - 2\,\zeta(2)\frac{z^2}{\pi^2} - 2\,\zeta(4)\frac{z^4}{\pi^4} + O(z^6)$$

we easily prove Euler's results that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. We may go further, if we have a convenient servant to do the Taylor expansions for us:

```
print(Vec(1 - z*cos(z)/sin(z) + O(z^14))/2);
[1/6, 0, 1/90, 0, 1/945, 0, 1/9450, 0, 1/93555, 0, 691/638512875, 0]
```

shows the origin of Euler's prime 691 in the numerator of $\zeta(12)/\pi^{12}$.

2.3.2 $\zeta(4)$ from Fourier analysis

Here I shall be even sketchier, since the context is more strongly related to my undergraduate studies than to my school work. Joseph Fourier (1768–1830) provided a method for representing a function, say $f(x)$, on a finite interval, say $\pi \geq x \geq -\pi$, by an infinite series of sinusoidal functions, say $\cos(nx)$ and $\sin(nx)$, with coefficients that may be evaluated from integrals of the product of $f(x)$ and the sinusoidal functions. His method looks much prettier if we use Euler's famous formula

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z)$$

where $\mathrm{i}^2 = -1$. But don't be put off by "the square root of -1"; we shall not need it, to evaluate $\zeta(4)$.

There are not many questions on Fourier analysis that can be solved straightforwardly, under examination conditions. Normally, the student is asked to analyze a very simple function, say $f(x) = x^2$, for which the Fourier coefficients in the series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(\mathrm{i}nx) \tag{12}$$

may be computed fairly easily. Now suppose that the student obtains the correct Fourier coefficients, namely

$$a_n = \int_{-\pi}^{\pi} \frac{\exp(-\mathrm{i}nx)}{2\pi} f(x)\,\mathrm{d}x = \int_0^{\pi} \frac{\cos(nx)}{\pi} x^2 \mathrm{d}x = \begin{cases} \pi^2/3 & \text{for } n = 0 \\ 2(-1)^n/n^2 & \text{for } n \neq 0 \end{cases}$$

then s/he might be asked to substitute $x = 0$ in the Fourier series (12) and obtain, thereby, an evaluation of the Riemann eta value $\eta(2) = \sum_{n>0}(-1)^{n-1}/n^2 = \zeta(2)/2$. This too is rather easy: at $x = 0$ we obtain $0 = \pi^2/3 - 4\eta(2)$ from the Fourier series (12), using the result for the Fourier coefficients.

An interesting exam question might say: "use Parseval's theorem to evaluate $\zeta(4)$".

The theorem in question gives an integral of the square of any function $f(x)$ as the sum of the squares of its Fourier coefficients a_n:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

2.3.3 Problem 6: $\zeta(4)$ from Fourier and Parseval

Use the given Fourier series for x^2 and Parseval's theorem to prove that $\zeta(4) = \pi^4/90$.

2.4 The sunshine number $\zeta(3)$

We have seen that a black body of surface area A and absolute temperature T radiates energy at a rate

$$\frac{\text{energy}}{\text{time}} = I_4 \frac{(kT)^4 A}{h^3 c^2}$$

where

$$I_4 = \int_0^\infty \frac{2\pi x^3 \, \mathrm{d}x}{\exp(x) - 1} \; = \; 12\pi\zeta(4) \; = \; \frac{2\pi^5}{15} \; \approx \; 40.80$$

derives from an integral over the black-body spectrum.

Like all forms of electromagnetic radiation, light is emitted and absorbed as "bundles" of energy called photons. The rate at which a black body of surface area A and temperature T emits photons is

$$\frac{\text{photons}}{\text{time}} = I_3 \frac{(kT)^3 A}{h^3 c^2}$$

where

$$I_3 = \int_0^\infty \frac{2\pi x^2 \, \mathrm{d}x}{\exp(x) - 1} \; = \; 4\pi\zeta(3) \; \approx \; 15.11$$

2.4.1 How many photons in your oven?

$$\frac{\text{photons}}{\text{volume}} = 16\pi\zeta(3) \left(\frac{kT}{hc}\right)^3$$

where the constant is now $4I_3$ and the factor of 4 takes account of the fact that only $\frac{1}{2}$ of the photons are moving towards the nearest wall and these have an average value of $\frac{1}{2}$ for the cosine of the angle of incidence with which they hit the wall.

In the U.K. the formula for the absolute temperature of an oven at "gas mark" $M \geq 1$ is

$$T(M) = \left(273.15 + \frac{125}{9}(M - 1) + 135\right) \text{ K}$$

since $M = 1$ is defined as 135 degrees Celsius (identical to 275 degrees Fahrenheit) and then each new mark increases the temperature by 25 degrees Fahrenheit, i.e. by 125/9 degrees Celsius. The zero of the Celsius scale is defined as 273.15 K, which is very close to the melting point of pure ice at standard atmospheric pressure.

Let's set an empty 40-litre oven at gas mark 9 and assume that the walls and door are perfectly black, on the inside. After this oven has arrived at the design temperature, we switch off the power. Working out

$$\frac{kT(9)}{hc} = \frac{1.381 \times 10^{-23} \times 519.3}{6.626 \times 10^{-34} \times 2.998 \times 10^8} \text{ m}^{-1} = 3.61 \times 10^4 \text{ m}^{-1}$$

we expect to have

$$16\pi\zeta(3) \times (3.61 \times 10^4)^3 \times 0.040 = 1.14 \times 10^{14} \text{ photons}$$

inside the oven. When the oven eventually cools down to room temperature, with $T \approx 300$ K, the number will have decreased by a factor of $(300/519)^3 = 0.193$, so now there will be merely 22 trillion photons inside.

Now let's turn down the oven to 2.73 K.

2.5 How many photons in your Universe?

About 13.7 billion years ago, the whole of the Universe was at a temperature comparable to the present temperature of the surface of the Sun. Photons, electrons, protons (and other particles) were in constant collision. But then the expansion of the Universe (in fact the expansion of space itself) cooled things down and when the temperature fell below about 3000 K the photons ceased to interact significantly with matter. Since then, space has expanded by a factor of about 1000, in each of its three dimensions, and the cosmic background radiation that we readily detect nowadays has a black-body spectrum with a temperature of about 3 K. The expansion of space has stretched out the most probable wavelength, $\lambda = (2.898 \times 10^{-3}\ \mathrm{m\ K})/T$, taking it from the visible region, at the temperature of the surface of the Sun, to the microwave region, now. In the last few years we have come to understand a great deal about the hot Big Bang in which our Universe originated, by studying this cosmic microwave background radiation. Its present temperature is known rather precisely:

$$T_0 = (2.728 \pm 0.002)\ \mathrm{K}$$

where the subscript 0 means "now".

The whole of the Universe now has a background photon density

$$\frac{\text{cosmic photons}}{\text{volume of Universe}} = 16\pi\zeta(3)\left(\frac{kT_0}{hc}\right)^3 = 4.12 \times 10^8\ \mathrm{m}^{-3}. \tag{13}$$

What "volume" should we multiply by, to count the total number of photons? For all we know, the Universe might have an infinite volume. Yet we cannot see all of it. Let's consider just that volume

$$V = \frac{4\pi}{3}R_0^3$$

out to the present distance R_0 of the places from which we now receive light from the Big Bang. Further than that we cannot see. It is believed, nowadays, that we are allowed to use Euclidean geometry in this calculation. Space-time is curved but space seems not be, on a cosmic scale.

You might think that R_0/c ought to be close to $t_0 = 13.7 \times 10^9$ years, the present age of the Universe. In fact, it's not as simple as that: the curvature of space-time in the early Universe means that we must use the general theory of relativity. In the simplest current model,

$$\frac{R_0}{ct_0} = \frac{I_1(\Omega_{\Lambda,0})}{I_2(\Omega_{\Lambda,0})}$$

is the ratio of two integrals that depend on the crucial number $\Omega_{\Lambda,0} = 0.734 \pm 0.02$, which is the fraction of the present space-time curvature of the Universe that comes from the cosmological constant. (You may have heard this contribution referred to, rather unhelpfully, as "dark energy".) The integrals are

$$I_1(c) = \int_0^1 \frac{\mathrm{d}a}{\sqrt{a^4c + (1-c)a}}$$

$$I_2(c) \;=\; \int_0^1 \frac{a\,\mathrm{d}a}{\sqrt{a^4c+(1-c)a}} \;=\; \frac{1}{3\sqrt{c}}\ln\left(\frac{1+\sqrt{c}}{1-\sqrt{c}}\right)$$

and the first is too hard to do by hand. So let's ask our servant to do it:

```
default(realprecision, 5); c = 0.734; s = sqrt(c);
print(intnum(a=0,1, 1/sqrt(a^4*c+(1-c)*a)));
3.4700
print(log((1+s)/(1-s))/3/s);
0.99677
```

There are about are 3.156×10^7 seconds in a year. So light travels $3.156 \times 10^7 \times 2.998 \times 10^8 = 9.46 \times 10^{15}$ metres in a year. Hence I claim that there are about

$$\frac{4\pi}{3}\left(\frac{3.47}{0.997} \times 13.7 \times 10^9 \times 9.46 \times 10^{15}\right)^3 \times 4.12 \times 10^8 = 1.58 \times 10^{89} \text{ photons}$$

present within the horizon beyond which you cannot possibly look.

Please remember that we needed the Riemann zeta value

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569\ldots$$

to do this cosmic calculation.

2.6 How many neutrinos in your Universe?

In nuclear beta-decay a neutron turns into a proton. An electron and a anti-neutrino are ejected from the nucleus:

$$\mathrm{n} \longrightarrow \mathrm{p} + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$$

sharing the released energy. We call $\overline{\nu}_\mathrm{e}$ the "electron anti-neutrino". Other nuclei eject positrons and electron neutrinos:

$$\mathrm{p} \longrightarrow \mathrm{n} + \mathrm{e}^+ + \nu_\mathrm{e}$$

A second pair of neutrinos is produced by the decays

$$\begin{aligned} \mu^- &\longrightarrow \mathrm{e}^- + \nu_\mu + \overline{\nu}_\mathrm{e} \\ \mu^+ &\longrightarrow \mathrm{e}^+ + \overline{\nu}_\mu + \nu_\mathrm{e} \end{aligned}$$

which occur in the Earth's atmosphere, where muons are created by cosmic rays. A third pair, ν_τ and $\overline{\nu}_\tau$, is readily produce by the decays of "tau-leptons", $\tau^\pm$, at high energy physics laboratories. There seem to be no more.

There is a cosmic background of neutrinos from the hot Big Bang with a ratio

$$\frac{\text{neutrinos}}{\text{photons}} = 3\,\frac{J_3}{I_3}\left(\frac{T_\nu}{T_0}\right)^3 \tag{14}$$

of the neutrino and photon densities, where $T_\nu \approx 1.95$ K is the present temperature of the neutrino background and

$$I_3 = \int_0^\infty \frac{2\pi x^2\,\mathrm{d}x}{\exp(x)-1}$$

$$J_3 = \int_0^\infty \frac{2\pi x^2\,\mathrm{d}x}{\exp(x)+1}$$

with a smaller integral J_3 for the neutrinos, because they are particles that obey an "exclusion principle". Neutrinos, like electrons, utterly refuse to share quantum states with particles identical to themselves. (This rather selfish behaviour is fortunate for us, because it is the exclusion principle for electrons that is responsible for the wonderful chemistry revealed by the periodic table of the elements.)

The factor of 3 in the ratio (14) comes from the three "flavours" of neutrino: e, μ and τ. The next factor comes from the exclusion principle. In the simplest model, the final factor is a rational number: $(T_\nu/T_0)^3 = \frac{4}{11}$.

2.6.1 Problem 7: $\zeta(3)$ integral for cosmic neutrinos

Show that $J_3 = 3\pi\zeta(3)$ and hence that there are, on average, about 337 neutrinos, left over from the Big Bang, in each cubic centimetre of your Universe. [Neutrinos permeate matter with great ease. There may be more than 300 cosmic neutrinos in each cubic centimetre of your brain, as you solve this problem, along with higher energy neutrinos from local sources, such as the Sun and decays of muons in the Earth's atmosphere.]

3 Feynman's zeta values

3.1 Particles and fields

On January 25, 1947, the journal Nature published a letter from Donald Perkins [13] (né 1925), entitled "Nuclear disintegration by meson capture".

On December 30, 1947, the Physical Review received a one-page article by Julian Schwinger [15] (1918–1994) entitled "On quantum-electrodynamics and the magnetic moment of the electron".

Between January and December of that year came momentous discoveries of two new types of particle, muons and pions, and, quite independently, the stimulus for great leaps of understanding in the quantum field theory of electromagnetism, by Schwinger, Hans Bethe (1906–2005), Richard Feynman (1918–1988) and Freeman Dyson (né 1923).

3.1.1 Particles

Pions (or pi-mesons as they were called in 1947) are created when high-energy protons from distant regions of the Universe hit the Earth's atmosphere. A negatively charged pion decays to give (most often) a muon and a neutrino:

$$\pi^- \longrightarrow \mu^- + \overline{\nu}_\mu$$

The muon then decays to give an electron and a pair of neutrinos. The discovery of pions and muons, in 1947, by the tracks that they leave in photographic emulsion, was the beginning of the experimental side of my subject: particle physics. Since then many more types of particles have been found, in collisions at particle accelerators. Currently, experimenters at the large hadron collider in Geneva are searching for a theoretically expected particle, called the "Higgs boson".

3.1.2 Fields

James Clerk Maxwell (1831–1879) described the electromagnetic fields, **E** and **B**, that are created by, and in turn influence, charged particles. An example is the force between a pair of current carrying wires, which is used to define the Ampère (A), as a unit of electrical current, in terms of the definition of the Newton (N), as a unit of force. This definition appears in Maxwell's equations via a constant

$$\mu_0 \equiv \frac{4\pi}{10^7}\ \mathrm{N\ A^{-2}}$$

which appears in magnetic-field calculations. Using this value, we may define a very important dimensionless number

$$\alpha \equiv \frac{e^2 c}{2h}\mu_0 = \frac{1}{137.0359990\ldots}$$

called the fine structure constant. It is know to an accuracy of about one part in a billion. Here e^2 is the square of the electron's charge, c is the speed of light and h is Planck's constant. Physicists tend to remember the reciprocal of the value of α, which is close to 137.

The quantum field theory developed in the late 1940's, by Schwinger, Feynman and Dyson, combined Maxwell's classical theory of electromagnetism (in which μ_0 occurs) with Einstein's special theory of relativity (in which c is rather important) and earlier quantum theory (in which h is rather important).

Werner Heisenberg (1901–1976), Erwin Schrödinger (1887–1961) and Paul Dirac (1902–1984) had developed the theory of quantum mechanics, between 1925 and 1932. This confronts the fundamental unpredictability of individual events, at the deepest level of physics, and gives ways of calculating the probabilities of outcomes, in a large number of measurements. But it does not handle the creation and annihilation of particles. Schwinger and Feynman solved that problem, in the interactions of photons, electrons and positrons.

In the 1930's it was realized that a "photon field" was needed, to explain the creation of photons by the interactions of charged particles, and that an "electron field" was needed, to explain the creation of electrons and positrons by the photon field.

I shall try to indicate how the "sunshine" number $\zeta(3)$, and its cousins $\zeta(5)$, $\zeta(7)$, etc, appear in the application of quantum field theory to particle physics. But first, it would be a good idea to look at the result of a rather simpler calculation.

3.1.3 Electron-positron creation in pion decay

The positron, e^+, is the anti-particle of the electron, e^-, with exactly the same mass, m_e, and exactly the opposite charge, e.

The neutral pion π^0 usually decays into a pair of photons:

$$\pi^0 \longrightarrow \gamma + \gamma$$

where γ is the symbol that we use for a photon (sometimes called a gamma ray). In the rest frame of the pion, each photon carries away an energy $\frac{1}{2} m_{\pi^0} c^2$, where m_{π^0} is the mass of π^0.

In 1951, Richard Dalitz (1925–2006) used the quantum field theory of Feynman and Schwinger to predict another, rarer decay:

$$\pi^0 \longrightarrow \gamma + \mathrm{e}^- + \mathrm{e}^+$$

in which an electron-positron pair is created, in place of one of the photons. His predicted rate was a fraction [9]

$$\text{Dalitz pair probability} \;=\; \frac{\alpha}{\pi}\left(\frac{4}{3}\ln\left(\frac{m_{\pi^0}}{m_\mathrm{e}}\right) - \frac{7}{3}\right) \;=\; 1.185\%$$

of the total decays. This was a rather successful prediction. The best result from modern experiments is that $(1.198 \pm 0.032)\%$ of the decays of π^0 have the electron-positron pair predicted by Dalitz.

Dalitz had to do an integral to make his prediction, since the total energy released is here shared between 3 particles, in the final state, and we must integrate over all the possible ways of doing that. In any individual decay, we cannot say how the energy will be shared. We can predict only the probabilities of the various outcomes and check those predictions against a large number of measurements.

3.2 Magnetic moment of the electron

$$\frac{\text{magnetic moment}}{\text{Bohr magneton}} = 1 + A_1 \frac{\alpha}{\pi} + A_2 \left(\frac{\alpha}{\pi}\right)^2 + A_3 \left(\frac{\alpha}{\pi}\right)^3 + \ldots$$

where the Bohr magneton

$$\frac{eh}{4\pi m_{\rm e}} = 9.274 \times 10^{-24}\ \mathrm{J\ T^{-1}}$$

has the units of Joules per Tesla and is the value predicted by Dirac in 1928. In 1947, Schwinger found the first correction term, with [15]

$$A_1 = \frac{1}{2}.$$

In 1950, Robert Karplus and Norman Kroll claimed the value [10]

$$28\zeta(3) - 54\zeta(2)\ln(2) + \frac{125}{6}\zeta(2) - \frac{2687}{288} \;=\; -2.972604271\ldots$$

for the coefficient of the next correction. It turned out that they had made a mistake in this rather difficult calculation. The correct result [16, 14]

$$A_2 \;=\; \frac{3}{4}\zeta(3) - 3\zeta(2)\ln(2) + \frac{1}{2}\zeta(2) + \frac{197}{144} \;=\; -0.3284789655\ldots$$

was not obtained until 1957. Not until 1996 was the next coefficient [12]

$$\begin{aligned} A_3 \;&=\; -\frac{215}{24}\zeta(5) + \frac{83}{12}\zeta(3)\zeta(2) - \frac{13}{8}\zeta(4) - \frac{50}{3}U_{3,1} \\ &\quad + \frac{139}{18}\zeta(3) - \frac{596}{3}\zeta(2)\ln(2) + \frac{17101}{135}\zeta(2) + \frac{28259}{5184} \qquad (15) \\ &=\; 1.181241456\ldots \end{aligned}$$

found by Stefano Laporta and Ettore Remiddi. The irrational numbers appearing on the second line are those already seen in A_2. On the first line we see zeta values and a new number, namely the alternating double sum

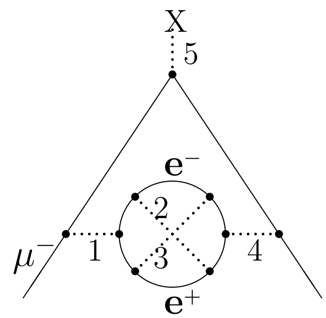
$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \approx -0.117875999650509326841013950834137618715 2\ldots$$

I visited Stefano and Ettore in Bologna when they were working on this formidable calculation and recommended to them a method of integration by parts, in D dimensions, that I had found useful for related calculations in the quantum field theory of electrons and photons. Here D is eventually set to 4, the number of dimensions of space-time. But it turns out to be easier if we keep it as a variable until the final stage of the calculation. Then if we find parts of the result with factors like $1/(D-4)$, or $1/(D-4)^2$, we need not worry that these parts tend to infinity as $D \to 4$; all that matters is that the complete result is finite. Based on my D-dimensional experience [6], I expected their final result to look simplest when written in terms of the new number $U_{3,1}$. This indeed turned out to be case. If, however, one writes (15) in terms of a less appropriate non-zeta value, such as [5]

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\left(\frac{1}{2}\right)^n = -\frac{1}{24}[\ln(2)]^4 + \frac{1}{4}\zeta(2)[\ln(2)]^2 + \frac{1}{4}\zeta(4) - \frac{1}{2}U_{3,1},$$

then the additional irrational numbers $[\ln(2)]^4$ and $\zeta(2)[\ln(2)]^2$ will also appear, since they are, as I had expected, absent when one uses $U_{3,1}$.

3.3 Feynman diagrams



Every Feynman diagram tells a possible story. In this one, a muon emits photon 1, which creates an electron and a positron. These exchange photons 2 and 3 and annihilate to give photon 4. Meanwhile, the muon interacts with a magnet X, via a (very low energy) photon 5, and then the muon neatly absorbs photon 4, to emerge from this story with (almost) the same energy and momentum as it started with.

This Feynman diagram specifies only one of the very large number of integrals that one must do to compute the term of order α^4 in the magnetic moment of the muon. Feynman found methods that give this particular contribution as an 8-dimensional integral.

In 1995, Pavel Baikov and I found a method of obtaining a numerical accuracy of 5 significant figures for such integrals [1]. A previous method, from Toichiro (Tom) Kinoshita [11], was shown to give errors of 5%.

Tom is one of my true heros. He was born in 1925 and studied with Sin-Itiro Tomonaga (1906–1979), who shared the 1965 Nobel Prize in physics with Schwinger and Feynman.

3.4 Multiple zeta values

When there is a single very large external energy scale, it is often a good approximation to neglect the masses of particles inside a Feynman diagram. This occurs in the case of electron-positron collisions at very high energy, which produce large numbers of particles, such as protons and pions, containing light quarks. So far all the results obtained from such diagrams have been zeta-valued, with $\zeta(7)$ now appearing in recent results [2].

I believe that this cannot continue, indefinitely. In 1995, Dirk Kreimer and I calculated some diagrams in a "toy" theory that has only one type of interacting quantum field, unlike the more pertinent electron-positron-photon and quark-antiquark-gluon theories of the electromagnetic and strong interactions. In this toy theory, we found Feynman diagrams that give a *multiple* zeta value [8, 7, 4]

$$\zeta(5,3) = \sum_{m>n>0} \frac{1}{m^5 n^3}$$

which is the first double sum that cannot be reduced to zeta values.

So I end with the conjecture that when practical quantum field theory progresses just one more step in difficulty, to calculations that yield the zeta value $\zeta(9)$, then those same calculations will also yield the multiple zeta value $\zeta(5,3)$.

4 Solutions

4.1 Solution 1

Let $f(x) = \frac{1}{30}x(x-1)(2x-1)(3x^2-3x-1)$. Then simple algebra shows that $f(x+1) - f(x) = x^4$. Hence it follows, by induction, that $\sum_{N>n>0} n^4 = f(N)$, for each integer $N \geq 2$, since this is true for $N = 2$,

4.2 Solution 2

We replace n by $p-n$ in the summation $S(p) = \sum_{p>n>0} 1/n^3$, add the result to $S(p)$ and divide by the prime p to obtain $2S(p)/p = \sum_{p>n>0} s(p,n)$, where

$$s(p,n) = \frac{1}{p}\left(\frac{1}{n^3} + \frac{1}{(p-n)^3}\right) = \frac{p^2 - 3pn + 3n^2}{n^3(p-n)^3} \equiv -\frac{3}{n^4} \pmod{p}.$$

Hence $2S(p) \equiv 0 \pmod{p}$ and $2S(p)/p \equiv -3\sum_{p>n>0} 1/n^4 \pmod{p}$. For each integer n with $p > n > 0$ there is a unique integer m with $p > m > 0$ and $m \equiv 1/n \pmod{p}$. Hence we replace n by m and obtain $2S(p)/p \equiv -3\sum_{p>m>0} m^4 \pmod{p}$. Finally we use the result (2) of Problem 1, which shows that $30\sum_{p>m>0} m^4 \equiv 0 \pmod{p}$ and hence that $20S(p)/p \equiv 0 \pmod{p}$. Thus the numerator of $S(p)$ is divisible by p^2 for every odd prime p that is not equal to 5.

4.3 Solution 3

For $|z| < 1$ we may use the binomial expansion $1/(1-z) = \sum_{n>0} z^{n-1}$. With $z = xy$ under the integral sign in (3), this gives $\sum_{n>0} 1/n^2 = \zeta(2)$, as claimed. With $z = -xy$ in (4), it gives $\sum_{n>0}(-1)^{n-1}/n^2$. Call this latter sum $\eta(2)$ (a Riemann eta value). Then $\zeta(2) - \eta(2) = 2\sum_{k>0} 1/(2k)^2 = \frac{1}{2}\zeta(2)$ and hence $\eta(2) = \frac{1}{2}\zeta(2)$, as claimed in (4). Now we compute the Jacobian in (9), which gives

$$\begin{aligned} J(x,y) &= \frac{\cos(a)}{\cos(b)}\frac{\cos(b)}{\cos(a)} - \frac{\sin(a)\sin(b)}{\cos^2(a)}\frac{\sin(a)\sin(b)}{\cos^2(b)} \\ &= 1 - \tan^2(a)\tan^2(b) = 1 - x^2y^2. \end{aligned}$$

Hence we may replace $\mathrm{d}x\,\mathrm{d}y/(1-x^2y^2)$ in (5) by the infinitesimal product $\mathrm{d}a\,\mathrm{d}b$. Finally we have to take care of the limits for a and b. We must impose the condition

$a+b \le \frac{\pi}{2}$ to ensure that neither x nor y exceeds 1. Hence we obtain half of the area of a square with side $\frac{\pi}{2}$, in (8), which proves that $\zeta(2) = \pi^2/6$.

[This is called an "elementary" proof. As you have seen, this does not mean that it is an "easy" proof. Rather, it means that the proof does not rely on some deep result in complex analysis. Later, you will see proofs that may appear to be much easier, yet rely on much deeper assumptions. When Sherlock Holmes (in the movies, if not in Conan Doyle's books) declares that his line of reasoning is "elementary, my dear Watson", he does not mean that it is trivial; rather he means that he can explain it, sometimes at length, without relying on external authority. The same type of thing occurs in mathematics. Frits Beukers, Eugenio Calabi and Johan Kolk published this "elementary" proof in 1993 [3].]

4.4 Solution 4

For $\zeta(12)$, both PSLQ and LLL should have given you Euler's result

$$\zeta(12) = \frac{691\pi^{12}}{638512875}$$

with the intriguing prime 691 in the numerator. You should not, according to current belief, have found any significant integer relation between the constants $[\zeta(13), \pi^{13}, 1]$. If you believe that you did, then please check it at higher precision, where it will almost certainly evaporate.

4.5 Solution 5

The black-body function may be written as $B(x) = 2\pi x^3 \sum_{n>0} \exp(-nx)$. To evaluate $I_4 = \int_0^\infty B(x)\,\mathrm{d}x$ we make the substitution $x = y/n$, in the n-th term, obtaining $I_4 = 2\pi\Gamma(4)\zeta(4)$, with the $1/n^4$ term in $\zeta(4) = \sum_{n>0} 1/n^4$ coming from the change of variables and an overall factor of

$$\Gamma(z) = \int_0^\infty y^{z-1} \exp(-y)\,\mathrm{d}y$$

appearing at $z = 4$. Integration by parts gives $\Gamma(z) = (z-1)\Gamma(z-1)$. Moreover, $\Gamma(1) = 1$. Hence, for positive integer n, we obtain $\Gamma(n) = (n-1)!$ after $n-1$ integrations by parts. Thus $I_4 = 12\pi\zeta(4)$.

4.6 Solution 6

With $f(x) = x^2$, Parseval's theorem gives $\pi^4/5 = (\pi^2/3)^2 + 8\zeta(4)$ and hence $\zeta(4) = (1/5 - 1/9)\pi^4/8 = \pi^4/90$.

4.7 Solution 7

The expansion of $1/(\exp(x)+1) = \sum_{n>0}(-1)^{n-1}\exp(-nx)$ gives a Riemann eta value in $J_3 = 4\pi\eta(3)$. Subtracting $\eta(3) = \sum_{n>0}(-1)^{n-1}/n^3$ from $\zeta(3) = \sum_{n>0} 1/n^3$, we obtain

$$\zeta(3) - \eta(3) = 2\sum_{k=1}^{\infty}\frac{1}{(2k)^3} = \frac{1}{4}\zeta(3).$$

Thus $J_3/I_3 = \eta(3)/\zeta(3) = \frac{3}{4}$ and the neutrino to photon ratio is $3\times\frac{3}{4}\times\frac{4}{11} = \frac{9}{11}$. We know from (12) that there are about 412 photons per cubic centimetre. Multiplying by $\frac{9}{11}$ we obtain 337 neutrinos per c.c.

5 Acknowledgments

5.1 The Manchester Grammar School: 1958–1964

I learnt calculus from Eric Hodge (1904–1962) who sadly died at the end of my fourth-form year. My enthusiasm for the method of induction comes from him. The somewhat awesome High Master of MGS at that time was Eric James (1909–1992) who had taught Freeman Dyson at Winchester. Dyson's latest book, *The Scientist as Rebel* (2006), is dedicated to Eric and Cordelia James.

My attention to detail in mathematical calculations was nurtured in the sixth form by Philip Scofield (1930–2010) for whom a sharpened piece of chalk was the most precise tool in all analysis. My physics teacher Aden Womersley (1931–2007) had a lively interest in modern developments and knew about quarks as soon as Gell–Mann hypothesized them in 1964. Yet my most influential teacher was John Scobell Armstrong (1927–2001) whose love of enlightenment philosophy spoke to both my heart and my head.

Around the time of the Cuban missile crisis of late 1962, as I recall, I became fascinated by the large factor $2\pi^5/15 \approx 40.80$ in the Stefan–Boltzmann constant that determines how much sunshine hits the upper atmosphere of planet Earth. Trying to understand its origin seemed like a good diversion from wondering whether Kennedy and Khrushchev were about to release catastrophic amounts of nuclear energy.

5.2 Zambia: 1965

At the end of 1964, I left the Manchester Grammar School and taught for 9 months at Munali Secondary School, in Lusaka. The discipline of teaching physics and maths to bright and highly motivated students did me a power of good. One of my former students, the late Dr Wedson C. Mwambazi, became the World Health Organization representative for Tanzania, and later for Ethiopia.

5.3 Oxford: 1965–1968

I was very fortunate to have Heinrich Kuhn (1904–1994), David Brink (né 1930) and Patrick Sandars (né 1935) as my physics tutors at Balliol College. From Don Perkins' lectures I discovered particle physics and became determined to study this subject as a post-graduate student at the newly established University of Sussex. I also heard in Oxford how seriously Dick Dalitz regarded the quark model.

5.4 Sussex: 1968–1971

I was equally fortunate in my postgraduate teachers, Gabriel Barton, David Bailin and Norman Dombey. It was fun, in my first term, to work out how Schwinger had obtained the correction term $\frac{1}{2}(\alpha/\pi)$ for the magnetic moment of the electron, back in 1947, the year of my birth. Stan Brodsky visited Sussex and told me how amazingly difficult it would be to obtain an exact value for the coefficient of $(\alpha/\pi)^3$.

5.5 Stanford: 1971–1973

This good fortune continued during my post-doctoral years. The Stanford Linear Accelerator Center (SLAC) in California proved just the right place to be, when physicists were discovering the "parts" of the proton, by scattering electrons off protons. Sid Drell and Jim Bjorken were great enthusiasts of Feynman's "parton model". Ken Johnson (1931–1999) was visiting SLAC. He was already sure that quantum field theory had come back to stay and would soon make sense of the strong interactions between pions and protons.

Yet it was also interesting to hear of competing metaphysics, from Geoff Chew at Berkeley, across the Bay. Geoff thought that there might be *no* quantum field theory of strong interactions. Instead, he advocated a "bootstrap" approach that might free us from previous reliance on the "undemocratic" supposition of forces between a few types of fundamental particle, somewhat in the manner of Baron von Münchhausen, who by legend tried to defy gravity by pulling himself up by his own boot laces.

At Stanford, however, we desperately wanted Feynman's partons to have the fractional charges and the "three mathematical colours" that had notionally been assigned to quarks. Those were the glory days, when experiment was the ruler of our subject, and we had to wait but a short while for extra data, from electron-positron collisions and neutrino-proton collisions, to confirm the quark-parton model and open the flood gates to a new quantum field theory.

5.6 Geneva: 1973–1974

At the laboratory of the European Council for Nuclear Reseach, CERN, in Geneva, I learnt from Gerard 't Hooft how quantum field theory could also make sense of the weak interaction, responsible for muon decay. Gerhard helped me to understand why strong interactions get weaker and weak interactions get stronger, at higher energies. Ken Johnson was right: quantum field theory is definitely here to stay.

I came to appreciate just how much hard work there was to do to meet the computational challenges of computing the Feynman diagrams of these new theories. It was Gerhard's supervisor, Martinus (Tiny) Veltman, who had begun to confront these challenges in the early 1970's. In 1974 there appeared a review of Tiny's computer-algebra package, `Schoonship`, whose modern-day successor `Form` has been developed by Jos Vermaseren to be a powerful engine which implements the subtle algorithms that yield zeta-values like $\zeta(7)$ in current calculations of massless Feynman diagrams.

5.7 Oxford: 1974–1975

Returning to Oxford as a junior research fellow, I revelled, with Dick Dalitz, as wonderful new results from electron-positron annihilation came from Stanford, and other laboratories, making the case for quarks more and more compelling.

One day, when I was scheduled to give a colloquium on these new results, a schoolboy from Eton came to grill me. His grasp of modern physics was remarkable and I wondered what he might later contribute to my subject. In fact, after a brief but distinguished period of work, Stephen Wolfram forsook particle physics. His programme `Mathematica` has been marketed with great commercial vigour and is now widely used by students and researchers to perform some of the simpler computations in particle physics.

5.8 The Open University: 1975–present

When I joined the Open University, one of my first tasks was to study Steven Weinberg's fine book, *Gravitation and Cosmology*, and then help to write a course, *Understanding Space and Time*. Julian Schwinger came to Milton Keynes, to work with us on producing this course. At my earnest request, he gave us a memorable 3-hour extempore account of general relativity, as he perceived it. I could scarcely believe my luck. I was getting paid to learn things, for myself, and then was able to help to share those good things with open-minded and enthusiastic students. My own openness of mind is reinforced by my avocation, by my partner Margaret and by our sons Stephen and Peter.

5.9 Envoi

To all those acknowledged above, and to many more, I give humble and hearty thanks for all their goodness and loving kindness.

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